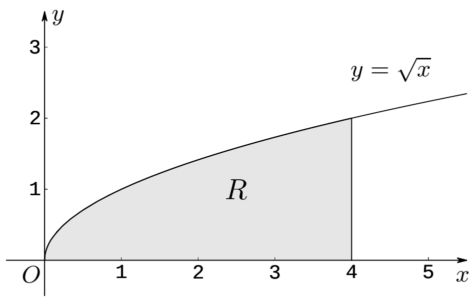
AP Calculus AB Free Response Notebook

Table of Contents

1998 AP Calculus AB Scoring Guidelines

1. Let R be the region bounded by the x–axis, the graph of $y = \sqrt{x}$, and the line $x = 4$.
   (a) Find the area of the region R.
   (b) Find the value of h such that the vertical line $x = h$ divides the region R into two regions of equal area.
   (c) Find the volume of the solid generated when R is revolved about the x–axis.
   (d) The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x–axis, they generate solids with equal volumes. Find the value of k.

(a)



$$A = \int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3} \quad \text{or} \quad 5.333$$

2 { 1: $A = \int_0^4 \sqrt{x}\,dx$; 1: answer }

(b) $\displaystyle\int_0^h \sqrt{x}\,dx = \frac{8}{3}$ –or– $\displaystyle\int_0^h \sqrt{x}\,dx = \int_h^4 \sqrt{x}\,dx$

$\displaystyle\frac{2}{3}h^{3/2} = \frac{8}{3}$ –or– $\displaystyle\frac{2}{3}h^{3/2} = \frac{16}{3} - \frac{2}{3}h^{3/2}$

$h = \sqrt[3]{16}$ or 2.520 or 2.519

2 { 1: equation in h; 1: answer }

(c) $\displaystyle V = \pi\int_0^4 (\sqrt{x})^2\,dx = \pi\frac{x^2}{2}\Big|_0^4 = 8\pi$

or 25.133 or 25.132

3 { 1: limits and constant; 1: integrand; 1: answer }

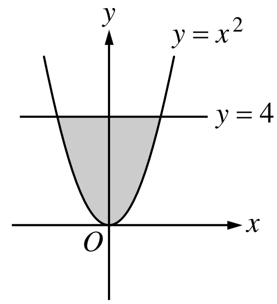
(d) $\displaystyle\pi\int_0^k (\sqrt{x})^2\,dx = 4\pi$ –or– $\displaystyle\pi\int_0^k (\sqrt{x})^2\,dx = \pi\int_k^4 (\sqrt{x})^2\,dx$

$\displaystyle\pi\frac{k^2}{2} = 4\pi$ –or– $\displaystyle\pi\frac{k^2}{2} = 8\pi - \pi\frac{k^2}{2}$

$k = \sqrt{8}$ or 2.828

2 { 1: equation in k; 1: answer }

2. The shaded region, R, is bounded by the graph of $y = x^2$ and the line $y = 4$, as shown in the figure above.
   (a) Find the area of R.
   (b) Find the volume of the solid generated by revolving R about the x–axis.
   (c) There exists a number k, $k > 4$, such that when R is revolved about the line $y = k$, the resulting solid has the same volume as the solid in part (b). Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.



(a) $$\text{Area} = \int_{-2}^{2} (4 - x^2)\,dx = 2\int_0^2 (4 - x^2)\,dx = 2\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{32}{3} = 10.666 \text{ or } 10.667$$

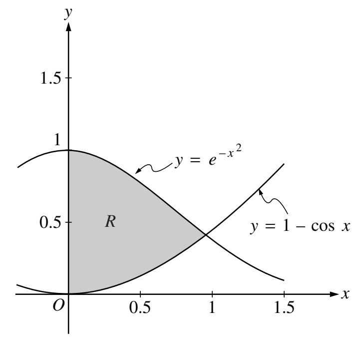
2 { 1: integral; 1: answer }

(b) $$\text{Volume} = \pi\int_{-2}^{2} \left(4^2 - (x^2)^2\right)\,dx = 2\pi\int_0^2 (16 - x^4)\,dx = 2\pi\left[16x - \frac{x^5}{5}\right]_0^2 = \frac{256\pi}{5} = 160.849 \text{ or } 160.850$$

3 { 1: limits and constant; 1: integrand; 1: answer }

(c) $$\pi\int_{-2}^{2} \left[(k - x^2)^2 - (k - 4)^2\right]\,dx = \frac{256\pi}{5}$$

4 { 1: limits and constant; 2: integrand < −1 > each error; 1: equation }



Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the y-axis, as shown in the figure above.

(a) Find the area of the region R.

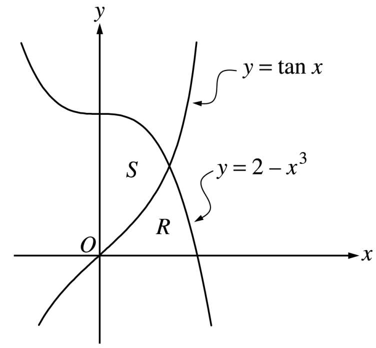
(b) Find the volume of the solid generated when the region R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

Region R $e^{-x^2} = 1 - \cos x$ at $x = 0.941944 = A$	1 : Correct limits in an integral in (a), (b), or (c).
(a) Area $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right) dx$ $= 0.590$ or 0.591	2 { 1 : integrand 1 : answer
(b) Volume $= \pi \int_0^A \left(\left(e^{-x^2}\right)^2 - (1 - \cos x)^2 \right) dx$ $= 0.55596\pi = 1.746$ or 1.747	3 { 2 : integrand and constant < −1> each error 1 : answer
(c) Volume $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right)^2 dx$ $= 0.461$	3 { 2 : integrand < −1> each error Note: 0/2 if not of the form $k \int_c^d (f(x) - g(x))^2 \, dx$ 1 : answer

Question 1

Let R and S be the regions in the first quadrant shown in the figure above. The region R is bounded by the x-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$. The region S is bounded by the y-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$.



(a) Find the area of R.

(b) Find the area of S.

(c) Find the volume of the solid generated when S is revolved about the x-axis.

Point of intersection

$2 - x^3 = \tan x$ at $(A, B) = (0.902155, 1.265751)$

(a) Area $R = \int_0^A \tan x \, dx + \int_A^{\sqrt[3]{2}} \left(2 - x^3\right) dx = 0.729$

or

Area $R = \int_0^B \left((2 - y)^{1/3} - \tan^{-1} y\right) dy = 0.729$

or

Area $R = \int_0^{\sqrt[3]{2}} \left(2 - x^3\right) dx - \int_0^A \left(2 - x^3 - \tan x\right) dx = 0.729$

3 : { 1 : limits; 1 : integrand; 1 : answer }

(b) Area $S = \int_0^A \left(2 - x^3 - \tan x\right) dx = 1.160$ or 1.161

or

Area $S = \int_0^B \tan^{-1} y \, dy + \int_B^2 (2 - y)^{1/3} \, dy = 1.160$ or 1.161

or

Area S

$= \int_0^2 (2 - y)^{1/3} \, dy - \int_0^B \left((2 - y)^{1/3} - \tan^{-1} y\right) dy$

$= 1.160$ or 1.161

3 : { 1 : limits; 1 : integrand; 1 : answer }

(c) Volume $= \pi \int_0^A \left(\left(2 - x^3\right)^2 - \tan^2 x\right) dx$

$= 2.652\pi$ or 8.331 or 8.332

3 : { 1 : limits and constant; 1 : integrand; 1 : answer }

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2002 SCORING GUIDELINES (Form B)

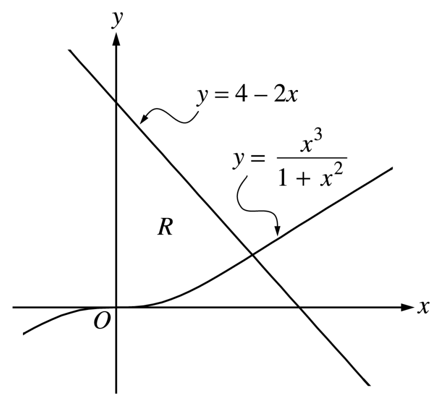
Question 1

Let R be the region bounded by the y-axis and the graphs of $y = \dfrac{x^3}{1+x^2}$ and $y = 4 - 2x$, as shown in the figure above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.



Region R

$$\frac{x^3}{1+x^2} = 4 - 2x \text{ at } x = 1.487664 = A$$

1 : Correct limits in an integral in (a), (b), or (c).

(a) Area $= \displaystyle\int_0^A \left(4 - 2x - \frac{x^3}{1+x^2}\right) dx$

$= 3.214$ or 3.215

2 $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) Volume

$$= \pi \int_0^A \left((4-2x)^2 - \left(\frac{x^3}{1+x^2}\right)^2 \right) dx$$

$= 31.884$ or 31.885 or 10.149π

3 $\begin{cases} 2 : \text{integrand and constant} \\ \quad < -1 > \text{ each error} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \displaystyle\int_0^A \left(4 - 2x - \frac{x^3}{1+x^2}\right)^2 dx$

$= 8.997$

3 $\begin{cases} 2 : \text{integrand} \\ \quad < -1 > \text{ each error} \\ \quad \text{note: 0/2 if not of the form} \\ \qquad k\displaystyle\int_c^d (f(x) - g(x))^2\, dx \\ 1 : \text{answer} \end{cases}$

AP® CALCULUS AB 2002 SCORING GUIDELINES

Question 1

Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.

(a) Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.

(b) Find the volume of the solid generated when the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.

(c) Let h be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$, and find the absolute maximum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$. Show the analysis that leads to your answers.

(a) Area $= \int_{1/2}^{1} (e^x - \ln x)\,dx = 1.222$ or 1.223

2 { 1 : integral; 1 : answer }

(b) Volume $= \pi \int_{1/2}^{1} \left((4 - \ln x)^2 - (4 - e^x)^2 \right) dx$

$= 7.515\pi$ or 23.609

4 {
1 : limits and constant
2 : integrand
< −1 > each error
Note: 0/2 if not of the form
$k \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$
1 : answer
}

(c) $h'(x) = f'(x) - g'(x) = e^x - \frac{1}{x} = 0$

$x = 0.567143$

Absolute minimum value and absolute maximum value occur at the critical point or at the endpoints.

$h(0.567143) = 2.330$
$h(0.5) = 2.3418$
$h(1) = 2.718$

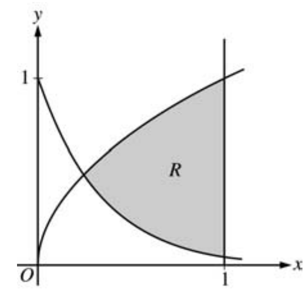
The absolute minimum is 2.330.
The absolute maximum is 2.718.

3 {
1 : considers $h'(x) = 0$
1 : identifies critical point and endpoints as candidates
1 : answers
}

Note: Errors in computation come off the third point.

Question 1

Let R be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.



(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 1$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a rectangle whose height is 5 times the length of its base in region R. Find the volume of this solid.

Point of intersection

$e^{-3x} = \sqrt{x}$ at $(T, S) = (0.238734, 0.488604)$

1: Correct limits in an integral in (a), (b), or (c)

(a) Area $= \int_T^1 \left(\sqrt{x} - e^{-3x}\right) dx$

$= 0.442$ or 0.443

$2: \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_T^1 \left(\left(1 - e^{-3x}\right)^2 - \left(1 - \sqrt{x}\right)^2 \right) dx$

$= 0.453\pi$ or 1.423 or 1.424

$3: \begin{cases} 2 : \text{integrand} \\ \quad < -1 > \ \text{reversal} \\ \quad < -1 > \ \text{error with constant} \\ \quad < -1 > \ \text{omits 1 in one radius} \\ \quad < -2 > \ \text{other errors} \\ 1 : \text{answer} \end{cases}$

(c) Length $= \sqrt{x} - e^{-3x}$

Height $= 5\left(\sqrt{x} - e^{-3x}\right)$

Volume $= \int_T^1 5\left(\sqrt{x} - e^{-3x}\right)^2 dx = 1.554$

$3: \begin{cases} 2 : \text{integrand} \\ \quad < -1 > \ \text{incorrect but has } \sqrt{x} - e^{-3x} \text{ as a factor} \\ 1 : \text{answer} \end{cases}$

Question 1

Let R be the region enclosed by the graph of $y = \sqrt{x-1}$, the vertical line $x = 10$, and the x-axis.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 3$.

(c) Find the volume of the solid generated when R is revolved about the vertical line $x = 10$.

(a) Area $= \int_1^{10} \sqrt{x-1}\, dx = 18$

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_1^{10} \left(9 - \left(3 - \sqrt{x-1}\right)^2\right) dx$
$= 212.057$ or 212.058

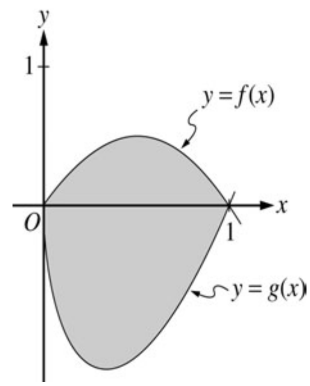
$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \pi \int_0^3 \left(10 - \left(y^2 + 1\right)\right)^2 dy$
$= 407.150$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

Question 2

Let f and g be the functions given by $f(x) = 2x(1 - x)$ and $g(x) = 3(x - 1)\sqrt{x}$ for $0 \le x \le 1$. The graphs of f and g are shown in the figure above.



(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Find the volume of the solid generated when the shaded region enclosed by the graphs of f and g is revolved about the horizontal line $y = 2$.

(c) Let h be the function given by $h(x) = kx(1 - x)$ for $0 \le x \le 1$. For each $k > 0$, the region (not shown) enclosed by the graphs of h and g is the base of a solid with square cross sections perpendicular to the x-axis. There is a value of k for which the volume of this solid is equal to 15. Write, but do not solve, an equation involving an integral expression that could be used to find the value of k.

(a) Area $= \int_0^1 (f(x) - g(x))\,dx$

$$= \int_0^1 \left(2x(1-x) - 3(x-1)\sqrt{x}\right) dx = 1.133$$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_0^1 \left((2 - g(x))^2 - (2 - f(x))^2\right) dx$

$$= \pi \int_0^1 \left(\left(2 - 3(x-1)\sqrt{x}\right)^2 - (2 - 2x(1-x))^2\right) dx$$

$= 16.179$

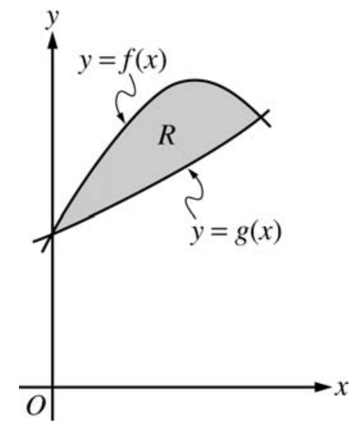
4 :
- 1 : limits and constant
- 2 : integrand
  - $\langle -1 \rangle$ each error
  - Note: $0/2$ if integral not of form $c\int_a^b \left(R^2(x) - r^2(x)\right) dx$
- 1 : answer

(c) Volume $= \int_0^1 (h(x) - g(x))^2\,dx$

$$\int_0^1 \left(kx(1-x) - 3(x-1)\sqrt{x}\right)^2 dx = 15$$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

Question 1



Let f and g be the functions given by $f(x) = 1 + \sin(2x)$ and $g(x) = e^{x/2}$. Let R be the shaded region in the first quadrant enclosed by the graphs of f and g as shown in the figure above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles with diameters extending from $y = f(x)$ to $y = g(x)$. Find the volume of this solid.

The graphs of f and g intersect in the first quadrant at $(S, T) = (1.13569, 1.76446)$.

1 : correct limits in an integral in (a), (b), or (c)

(a) $$\text{Area} = \int_0^S (f(x) - g(x))\,dx = \int_0^S \left(1 + \sin(2x) - e^{x/2}\right) dx = 0.429$$

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $$\text{Volume} = \pi\int_0^S \left((f(x))^2 - (g(x))^2\right) dx = \pi\int_0^S \left((1 + \sin(2x))^2 - \left(e^{x/2}\right)^2\right) dx = 4.266 \text{ or } 4.267$$

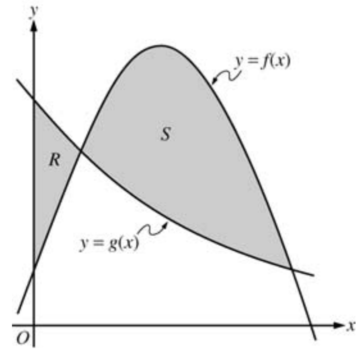
3 :
- 2 : integrand
  - ⟨−1⟩ each error
  - Note: 0/2 if integral not of form $c\int_a^b \left(R^2(x) - r^2(x)\right) dx$
- 1 : answer

(c) $$\text{Volume} = \int_0^S \frac{\pi}{2}\left(\frac{f(x) - g(x)}{2}\right)^2 dx = \int_0^S \frac{\pi}{2}\left(\frac{1 + \sin(2x) - e^{x/2}}{2}\right)^2 dx = 0.077 \text{ or } 0.078$$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

Question 1

Let f and g be the functions given by $f(x) = \frac{1}{4} + \sin(\pi x)$ and $g(x) = 4^{-x}$. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of f and g, and let S be the shaded region in the first quadrant enclosed by the graphs of f and g, as shown in the figure above.



(a) Find the area of R.

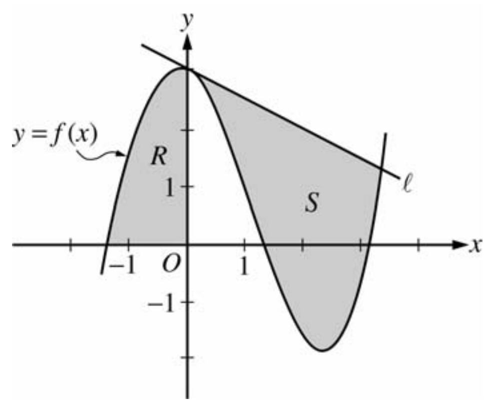
(b) Find the area of S.

(c) Find the volume of the solid generated when S is revolved about the horizontal line $y = -1$.

$f(x) = g(x)$ when $\frac{1}{4} + \sin(\pi x) = 4^{-x}$.
f and g intersect when $x = 0.178218$ and when $x = 1$.
Let $a = 0.178218$.

Solution	Scoring
(a) $\int_0^a (g(x) - f(x))\,dx = 0.064$ or 0.065	3 : { 1 : limits 1 : integrand 1 : answer
(b) $\int_a^1 (f(x) - g(x))\,dx = 0.410$	3 : { 1 : limits 1 : integrand 1 : answer
(c) $\pi \int_a^1 \left((f(x)+1)^2 - (g(x)+1)^2 \right) dx = 4.558$ or 4.559	3 : { 2 : integrand 1 : limits, constant, and answer

Question 1



Let f be the function given by $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$. Let R be the shaded region in the second quadrant bounded by the graph of f, and let S be the shaded region bounded by the graph of f and line ℓ, the line tangent to the graph of f at $x = 0$, as shown above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an integral expression that can be used to find the area of S.

For $x < 0$, $f(x) = 0$ when $x = -1.37312$.
Let $P = -1.37312$.

(a) Area of $R = \int_P^0 f(x)\,dx = 2.903$

2 : { 1 : integral
1 : answer

(b) Volume $= \pi\int_P^0 \left((f(x) + 2)^2 - 4 \right) dx = 59.361$

4 : { 1 : limits and constant
2 : integrand
1 : answer

(c) The equation of the tangent line ℓ is $y = 3 - \frac{1}{2}x$.

The graph of f and line ℓ intersect at $A = 3.38987$.

Area of $S = \int_0^A \left(\left(3 - \frac{1}{2}x\right) - f(x) \right) dx$

3 : { 1 : tangent line
1 : integrand
1 : limits

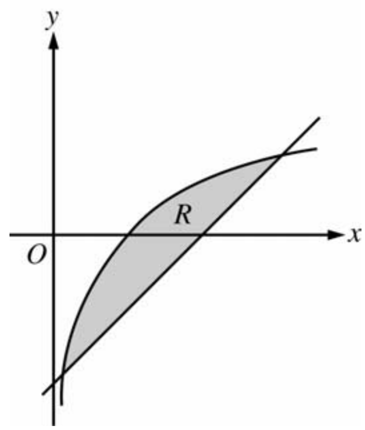
Question 1

Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$.

(c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the y-axis.



$\ln(x) = x - 2$ when $x = 0.15859$ and 3.14619.
Let $S = 0.15859$ and $T = 3.14619$

(a) Area of $R = \int_S^T (\ln(x) - (x - 2))\, dx = 1.949$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_S^T \left((\ln(x) + 3)^2 - (x - 2 + 3)^2 \right) dx$
$= 34.198$ or 34.199

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits, constant, and answer} \end{cases}$

(c) Volume $= \pi \int_{S-2}^{T-2} \left((y + 2)^2 - \left(e^y\right)^2 \right) dy$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

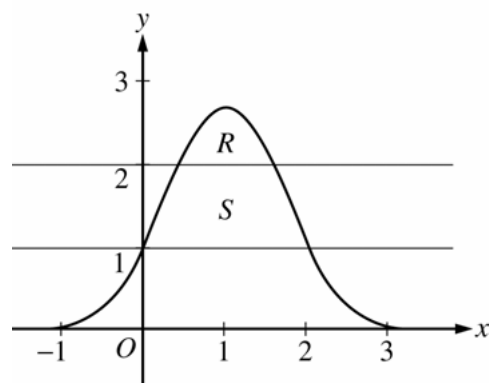
Question 1

Let R be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let S be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.

(a) Find the area of R.

(b) Find the area of S.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 1$.



$e^{2x-x^2} = 2$ when $x = 0.446057, 1.553943$
Let $P = 0.446057$ and $Q = 1.553943$

(a) Area of $R = \int_P^Q \left(e^{2x-x^2} - 2\right) dx = 0.514$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(b) $e^{2x-x^2} = 1$ when $x = 0,\ 2$

Area of $S = \int_0^2 \left(e^{2x-x^2} - 1\right) dx -$ Area of R

$= 2.06016 -$ Area of $R = 1.546$

OR

$\int_0^P \left(e^{2x-x^2} - 1\right) dx + (Q - P) \cdot 1 + \int_Q^2 \left(e^{2x-x^2} - 1\right) dx$

$= 0.219064 + 1.107886 + 0.219064 = 1.546$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \pi \int_P^Q \left(\left(e^{2x-x^2} - 1\right)^2 - (2-1)^2 \right) dx$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{constant and limits} \end{cases}$

Question 1

Let R be the region in the first and second quadrants bounded above by the graph of $y = \dfrac{20}{1+x^2}$ and below by the horizontal line $y = 2$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the x-axis.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles. Find the volume of this solid.

$\dfrac{20}{1+x^2} = 2$ when $x = \pm 3$	1 : correct limits in an integral in (a), (b), or (c)
(a) Area $= \displaystyle\int_{-3}^{3} \left(\frac{20}{1+x^2} - 2 \right) dx = 37.961$ or 37.962	2 : { 1 : integrand; 1 : answer
(b) Volume $= \pi \displaystyle\int_{-3}^{3} \left(\left(\frac{20}{1+x^2} \right)^2 - 2^2 \right) dx = 1871.190$	3 : { 2 : integrand; 1 : answer
(c) Volume $= \dfrac{\pi}{2} \displaystyle\int_{-3}^{3} \left(\frac{1}{2} \left(\frac{20}{1+x^2} - 2 \right) \right)^2 dx$ $= \dfrac{\pi}{8} \displaystyle\int_{-3}^{3} \left(\frac{20}{1+x^2} - 2 \right)^2 dx = 174.268$	3 : { 2 : integrand; 1 : answer

Question 1

Let R be the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$ and $y = \dfrac{x}{3}$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the vertical line $x = -1$.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the y-axis are squares. Find the volume of this solid.

The graphs of $y = \sqrt{x}$ and $y = \dfrac{x}{3}$ intersect at the points $(0, 0)$ and $(9, 3)$.

(a) $$\int_0^9 \left(\sqrt{x} - \frac{x}{3}\right) dx = 4.5$$

OR

$$\int_0^3 \left(3y - y^2\right) dy = 4.5$$

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $$\pi \int_0^3 \left((3y+1)^2 - \left(y^2+1\right)^2\right) dy = \frac{207\pi}{5} = 130.061 \text{ or } 130.062$$

$4 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $$\int_0^3 \left(3y - y^2\right)^2 dy = 8.1$$

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and answer} \end{cases}$

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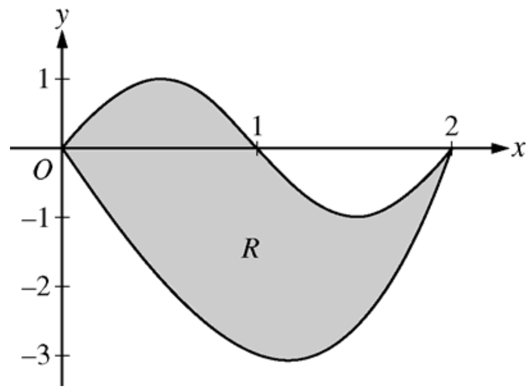
Question 1



Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$ Area $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$	3 : { 1 : limits 1 : integrand 1 : answer
(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$ The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$	2 : { 1 : limits 1 : integrand
(c) Volume $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 9.978$	2 : { 1 : integrand 1 : answer
(d) Volume $= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369$ or 8.370	2 : { 1 : integrand 1 : answer

1992 AB4/BC1

Consider the curve defined by the equation $y + \cos y = x + 1$ for $0 \le y \le 2\pi$.

(a) Find $\dfrac{dy}{dx}$ in terms of y.

(b) Write an equation for each vertical tangent to the curve.

(c) Find $\dfrac{d^2y}{dx^2}$ in terms of y.

1992 AB4/BC1
Solution

(a) $\frac{dy}{dx} - \sin y \frac{dy}{dx} = 1$

$\frac{dy}{dx}(1 - \sin y) = 1$

$\frac{dy}{dx} = \frac{1}{1 - \sin y}$

(b) $\frac{dy}{dx}$ undefined when $\sin y = 1$

$y = \frac{\pi}{2}$

$\frac{\pi}{2} + 0 = x + 1$

$x = \frac{\pi}{2} - 1$

(c) $$\frac{d^2 y}{dx^2} = \frac{d\left(\frac{1}{1 - \sin y}\right)}{dx}$$

$$= \frac{-\left(-\cos y \frac{dy}{dx}\right)}{(1 - \sin y)^2}$$

$$= \frac{\cos y\left(\frac{1}{1 - \sin y}\right)}{(1 - \sin y)^2}$$

$$= \frac{\cos y}{(1 - \sin y)^3}$$

1991 AB2

Let R be the region between the graphs of $y = 1 + \sin(\pi x)$ and $y = x^2$ from $x = 0$ to $x = 1$.

(a) Find the area of R .

(b) Set up, but do not integrate an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the *x*-axis.

(c) Set up, but do not integrate an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the *y*-axis.

1991 AB2
Solution

(a) $A = \int_0^1 1 + \sin(\pi x) - x^2 \, dx$

$= \left(x - \frac{1}{\pi}\cos(\pi x) - \frac{1}{3}x^3 \right)\Big|_0^1$

$= \left(1 - \frac{1}{\pi}(-1) - \frac{1}{3} \right) - \left(0 - \frac{1}{\pi} - 0 \right)$

$= \frac{2}{3} + \frac{2}{\pi}$

(b) $V = \pi \int_0^1 \left(1 + \sin(\pi x)\right)^2 - x^4 \, dx$

or

$2\pi \int_0^1 y^{3/2} \, dy + 2\pi \int_1^2 y\left(1 - \frac{2}{\pi}\arcsin(y-1) \right) dy$

(c) $V = 2\pi \int_0^1 x\left(1 + \sin(\pi x) - x^2\right) dx$

or

$\pi \int_0^1 y \, dy + \pi \int_1^2 \left(1 - \frac{1}{\pi}\arcsin(y-1) \right)^2 - \left(\frac{1}{\pi}\arcsin(y-1) \right)^2 dy$

1990 AB3

Let R be the region enclosed by the graphs of $\boldsymbol{y} = \boldsymbol{e}^{\boldsymbol{x}}$, $\boldsymbol{y} = (\boldsymbol{x}-1)^2$, and the line $\boldsymbol{x} = 1$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the y-axis.

1990 AB3
Solution

(a) $A = \int_0^1 e^x - (x-1)^2\, dx$

$= \int_0^1 e^x - x^2 + 2x - 1\, dx$

$= e^x \Big]_0^1 - \frac{1}{3}(x-1)^3 \Big]_0^1$

$= (e-1) - \frac{1}{3} = e - \frac{4}{3}$

(b) $V = \pi \int_0^1 e^{2x} - (x-1)^4\, dx$

$= \pi \left[\frac{e^{2x}}{2} \right] \Bigg|_0^1 - \pi \left[\frac{1}{5}(x-1)^5 \right] \Bigg|_0^1$

$= \pi \left[\left(\frac{e^2}{2} - \frac{1}{2} \right) - \frac{1}{5} \right] = \pi \left(\frac{e^2}{2} - \frac{7}{10} \right)$

or

$V = 2\pi \int_0^1 y\left[1 - \left(1 - \sqrt{y}\right)\right] dy + 2\pi \int_1^e y(1 - \ln y)\, dy$

$= 2\pi \cdot \frac{2}{5} y^{5/2} \Big|_0^1 + 2\pi \left[\frac{1}{2} y^2 - \left(\frac{1}{2} y^2 \ln y - \frac{1}{4} y^2 \right) \right] \Bigg|_1^e$

$= \frac{4}{5}\pi + 2\pi \left[\frac{1}{4} e^2 - \frac{3}{4} \right] = \pi \left(\frac{e^2}{2} - \frac{7}{10} \right)$

(c) $V = 2\pi \int_0^1 x\left[e^x - (x-1)^2 \right] dx$

or

$V = \pi \int_0^1 1 - \left(1 - \sqrt{y}\right)^2 dy + \pi \int_1^e 1 - (\ln y)^2\, dy$

1989 AB2

Let R be the region in the first quadrant enclosed by the graph of $y = \sqrt{6x + 4}$, the line $y = 2x$, and the y-axis.

(a) Find the area of R.

(b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the x-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the y-axis.

1989 AB2
Solution

(a) Area $= \int_0^2 \sqrt{6x+4} - 2x\,dx$

$$= \frac{1}{6}\cdot\frac{2}{3}\left(6x+4\right)^{3/2} - x^2\Big|_0^2$$

$$= \left(\frac{64}{9} - 4\right) - \frac{8}{9} = \frac{20}{9}$$

(b) Volume about x-axis

$$V = \pi\int_0^2 \left(6x+4\right) - 4x^2\,dx$$

<u>or</u>

$$V = \pi\int_0^2 \left(6x+4\right)dx - \frac{32\pi}{3}$$

(c) Volume about y-axis

$$V = 2\pi\int_0^2 x\left(\sqrt{6x+4} - 2x\right)dx$$

<u>or</u>

$$V = \pi\int_0^4 \left(\frac{y}{2}\right)^2 dy - \pi\int_2^4 \left(\frac{y^2-4}{6}\right)^2 dy$$

Question 2

The rate at which people enter an amusement park on a given day is modeled by the function E defined by

$$E(t) = \frac{15600}{\left(t^2 - 24t + 160\right)}.$$

The rate at which people leave the same amusement park on the same day is modeled by the function L defined by

$$L(t) = \frac{9890}{\left(t^2 - 38t + 370\right)}.$$

Both $E(t)$ and $L(t)$ are measured in people per hour and time t is measured in hours after midnight. These functions are valid for $9 \le t \le 23$, the hours during which the park is open. At time $t = 9$, there are no people in the park.

(a) How many people have entered the park by 5:00 P.M. ($t = 17$)? Round answer to the nearest whole number.

(b) The price of admission to the park is \$15 until 5:00 P.M. ($t = 17$). After 5:00 P.M., the price of admission to the park is \$11. How many dollars are collected from admissions to the park on the given day? Round your answer to the nearest whole number.

(c) Let $H(t) = \int_9^t (E(x) - L(x))\,dx$ for $9 \le t \le 23$. The value of $H(17)$ to the nearest whole number is 3725. Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the park.

(d) At what time t, for $9 \le t \le 23$, does the model predict that the number of people in the park is a maximum?

(a) $\int_9^{17} E(t)\,dt = 6004.270$

6004 people entered the park by 5 pm.

3: { 1: limits; 1: integrand; 1: answer }

(b) $15\int_9^{17} E(t)\,dt + 11\int_{17}^{23} E(t)\,dt = 104048.165$

The amount collected was \$104,048.

or

$\int_{17}^{23} E(t)\,dt = 1271.283$

1271 people entered the park between 5 pm and 11 pm, so the amount collected was

$\$15 \cdot (6004) + \$11 \cdot (1271) = \$104,041.$

1: setup

(c) $H'(17) = E(17) - L(17) = -380.281$

There were 3725 people in the park at $t = 17$.

The number of people in the park was decreasing at the rate of approximately 380 people/hr at time $t = 17$.

3: { 1: value of $H'(17)$; 2: meanings — 1: meaning of $H(17)$; 1: meaning of $H'(17)$; $< -1 >$ if no reference to $t = 17$ }

(d) $H'(t) = E(t) - L(t) = 0$

$t = 15.794$ or 15.795

2: { 1: $E(t) - L(t) = 0$; 1: answer }

Question 2

A tank contains 125 gallons of heating oil at time $t = 0$. During the time interval $0 \le t \le 12$ hours, heating oil is pumped into the tank at the rate

$$H(t) = 2 + \frac{10}{(1 + \ln(t+1))} \text{ gallons per hour.}$$

During the same time interval, heating oil is removed from the tank at the rate

$$R(t) = 12\sin\left(\frac{t^2}{47}\right) \text{ gallons per hour.}$$

(a) How many gallons of heating oil are pumped into the tank during the time interval $0 \le t \le 12$ hours?

(b) Is the level of heating oil in the tank rising or falling at time $t = 6$ hours? Give a reason for your answer.

(c) How many gallons of heating oil are in the tank at time $t = 12$ hours?

(d) At what time t, for $0 \le t \le 12$, is the volume of heating oil in the tank the least? Show the analysis that leads to your conclusion.

Solution	Scoring
(a) $\int_0^{12} H(t)\,dt = 70.570$ or 70.571	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(b) $H(6) - R(6) = -2.924$, so the level of heating oil is falling at $t = 6$.	1 : answer with reason
(c) $125 + \int_0^{12} (H(t) - R(t))\,dt = 122.025$ or 122.026	$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(d) The absolute minimum occurs at a critical point or an endpoint. $H(t) - R(t) = 0$ when $t = 4.790$ and $t = 11.318$. The volume increases until $t = 4.790$, then decreases until $t = 11.318$, then increases, so the absolute minimum will be at $t = 0$ or at $t = 11.318$. $125 + \int_0^{11.318} (H(t) - R(t))\,dt = 120.738$ Since the volume is 125 at $t = 0$, the volume is least at $t = 11.318$.	$3: \begin{cases} 1: \text{sets } H(t) - R(t) = 0 \\ 1: \text{volume is least at } t = 11.318 \\ 1: \text{analysis for absolute minimum} \end{cases}$

Question 2

For $0 \le t \le 31$, the rate of change of the number of mosquitoes on Tropical Island at time t days is modeled by $R(t) = 5\sqrt{t}\cos\left(\frac{t}{5}\right)$ mosquitoes per day. There are 1000 mosquitoes on Tropical Island at time $t = 0$.

(a) Show that the number of mosquitoes is increasing at time $t = 6$.

(b) At time $t = 6$, is the number of mosquitoes increasing at an increasing rate, or is the number of mosquitoes increasing at a decreasing rate? Give a reason for your answer.

(c) According to the model, how many mosquitoes will be on the island at time $t = 31$? Round your answer to the nearest whole number.

(d) To the nearest whole number, what is the maximum number of mosquitoes for $0 \le t \le 31$? Show the analysis that leads to your conclusion.

Solution	Scoring
(a) Since $R(6) = 4.438 > 0$, the number of mosquitoes is increasing at $t = 6$.	1 : shows that $R(6) > 0$
(b) $R'(6) = -1.913$ Since $R'(6) < 0$, the number of mosquitoes is increasing at a decreasing rate at $t = 6$.	2 : { 1 : considers $R'(6)$ 1 : answer with reason
(c) $1000 + \int_0^{31} R(t)\,dt = 964.335$ To the nearest whole number, there are 964 mosquitoes.	2 : { 1 : integral 1 : answer
(d) $R(t) = 0$ when $t = 0$, $t = 2.5\pi$, or $t = 7.5\pi$ $R(t) > 0$ on $0 < t < 2.5\pi$ $R(t) < 0$ on $2.5\pi < t < 7.5\pi$ $R(t) > 0$ on $7.5\pi < t < 31$ The absolute maximum number of mosquitoes occurs at $t = 2.5\pi$ or at $t = 31$. $1000 + \int_0^{2.5\pi} R(t)\,dt = 1039.357$, There are 964 mosquitoes at $t = 31$, so the maximum number of mosquitoes is 1039, to the nearest whole number.	4 : { 2 : absolute maximum value 1 : integral 1 : answer 2 : analysis 1 : computes interior critical points 1 : completes analysis

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Question 1

Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

(a) To the nearest whole number, how many cars pass through the intersection over the 30-minute period?

(b) Is the traffic flow increasing or decreasing at $t = 7$? Give a reason for your answer.

(c) What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

(d) What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

(a) $\int_0^{30} F(t)\,dt = 2474$ cars	3 : { 1 : limits 1 : integrand 1 : answer
(b) $F'(7) = -1.872$ or -1.873 Since $F'(7) < 0$, the traffic flow is decreasing at $t = 7$.	1 : answer with reason
(c) $\frac{1}{5}\int_{10}^{15} F(t)\,dt = 81.899$ cars/min	3 : { 1 : limits 1 : integrand 1 : answer
(d) $\frac{F(15) - F(10)}{15 - 10} = 1.517$ or 1.518 cars/min^2	1 : answer
Units of cars/min in (c) and cars/min^2 in (d)	1 : units in (c) and (d)

Question 2

A water tank at Camp Newton holds 1200 gallons of water at time $t = 0$. During the time interval $0 \le t \le 18$ hours, water is pumped into the tank at the rate

$$W(t) = 95\sqrt{t}\,\sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$R(t) = 275\sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

(a) Is the amount of water in the tank increasing at time $t = 15$? Why or why not?

(b) To the nearest whole number, how many gallons of water are in the tank at time $t = 18$?

(c) At what time t, for $0 \le t \le 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

(d) For $t > 18$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank becomes empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

(a) No; the amount of water is not increasing at $t = 15$ since $W(15) - R(15) = -121.09 < 0$.

1 : answer with reason

(b) $1200 + \int_0^{18} (W(t) - R(t))\,dt = 1309.788$

1310 gallons

3 : { 1 : limits; 1 : integrand; 1 : answer }

(c) $W(t) - R(t) = 0$

$t = 0, 6.4948, 12.9748$

t (hours)	gallons of water
0	1200
6.495	525
12.975	1697
18	1310

The values at the endpoints and the critical points show that the absolute minimum occurs when $t = 6.494$ or 6.495.

3 : { 1 : interior critical points; 1 : amount of water is least at $t = 6.494$ or 6.495; 1 : analysis for absolute minimum }

(d) $\int_{18}^{k} R(t)\,dt = 1310$

2 : { 1 : limits; 1 : equation }

Question 2

The tide removes sand from Sandy Point Beach at a rate modeled by the function R, given by

$$R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right).$$

A pumping station adds sand to the beach at a rate modeled by the function S, given by

$$S(t) = \frac{15t}{1 + 3t}.$$

Both $R(t)$ and $S(t)$ have units of cubic yards per hour and t is measured in hours for $0 \le t \le 6$. At time $t = 0$, the beach contains 2500 cubic yards of sand.

(a) How much sand will the tide remove from the beach during this 6-hour period? Indicate units of measure.

(b) Write an expression for $Y(t)$, the total number of cubic yards of sand on the beach at time t.

(c) Find the rate at which the total amount of sand on the beach is changing at time $t = 4$.

(d) For $0 \le t \le 6$, at what time t is the amount of sand on the beach a minimum? What is the minimum value? Justify your answers.

(a) $\int_0^6 R(t)\,dt = 31.815$ or 31.816 yd^3

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer with units} \end{cases}$

(b) $Y(t) = 2500 + \int_0^t (S(x) - R(x))\,dx$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(c) $Y'(t) = S(t) - R(t)$

$Y'(4) = S(4) - R(4) = -1.908$ or -1.909 yd^3/hr

1 : answer

(d) $Y'(t) = 0$ when $S(t) - R(t) = 0$.
The only value in $[0, 6]$ to satisfy $S(t) = R(t)$ is $a = 5.117865$.

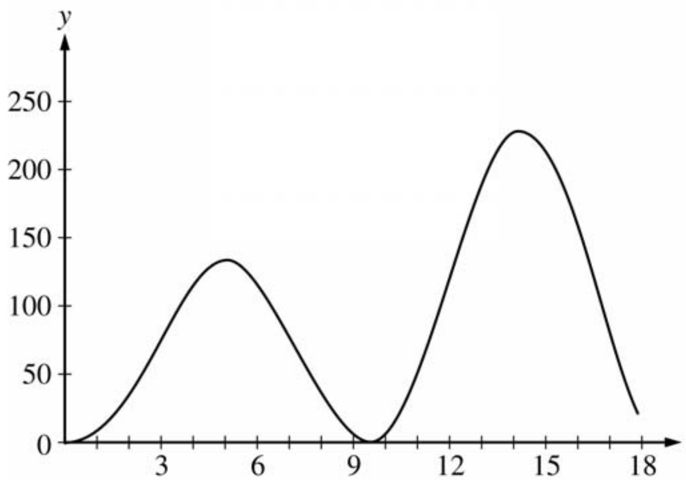
t	$Y(t)$
0	2500
a	2492.3694
6	2493.2766

The amount of sand is a minimum when $t = 5.117$ or 5.118 hours. The minimum value is 2492.369 cubic yards.

$3 : \begin{cases} 1 : \text{sets } Y'(t) = 0 \\ 1 : \text{critical } t\text{-value} \\ 1 : \text{answer with justification} \end{cases}$

Question 2

At an intersection in Thomasville, Oregon, cars turn left at the rate $L(t) = 60\sqrt{t}\sin^2\left(\frac{t}{3}\right)$ cars per hour over the time interval $0 \le t \le 18$ hours. The graph of $y = L(t)$ is shown above.



(a) To the nearest whole number, find the total number of cars turning left at the intersection over the time interval $0 \le t \le 18$ hours.

(b) Traffic engineers will consider turn restrictions when $L(t) \ge 150$ cars per hour. Find all values of t for which $L(t) \ge 150$ and compute the average value of L over this time interval. Indicate units of measure.

(c) Traffic engineers will install a signal if there is any two-hour time interval during which the product of the total number of cars turning left and the total number of oncoming cars traveling straight through the intersection is greater than 200,000. In every two-hour time interval, 500 oncoming cars travel straight through the intersection. Does this intersection require a traffic signal? Explain the reasoning that leads to your conclusion.

(a) $\int_0^{18} L(t)\,dt \approx 1658$ cars

$2 : \begin{cases} 1 : \text{setup} \\ 1 : \text{answer} \end{cases}$

(b) $L(t) = 150$ when $t = 12.42831,\ 16.12166$

Let $R = 12.42831$ and $S = 16.12166$

$L(t) \ge 150$ for t in the interval $[R, S]$

$\frac{1}{S-R}\int_R^S L(t)\,dt = 199.426$ cars per hour

$3 : \begin{cases} 1 : t\text{-interval when } L(t) \ge 150 \\ 1 : \text{average value integral} \\ 1 : \text{answer with units} \end{cases}$

(c) For the product to exceed 200,000, the number of cars turning left in a two-hour interval must be greater than 400.

$\int_{13}^{15} L(t)\,dt = 431.931 > 400$

$4 : \begin{cases} 1 : \text{considers 400 cars} \\ 1 : \text{valid interval } [h, h+2] \\ 1 : \text{value of } \int_h^{h+2} L(t)\,dt \\ 1 : \text{answer and explanation} \end{cases}$

OR

The number of cars turning left will be greater than 400 on a two-hour interval if $L(t) \ge 200$ on that interval.

$L(t) \ge 200$ on any two-hour subinterval of $[13.25304, 15.32386]$.

OR

$4 : \begin{cases} 1 : \text{considers 200 cars per hour} \\ 1 : \text{solves } L(t) \ge 200 \\ 1 : \text{discusses 2 hour interval} \\ 1 : \text{answer and explanation} \end{cases}$

Yes, a traffic signal is required.

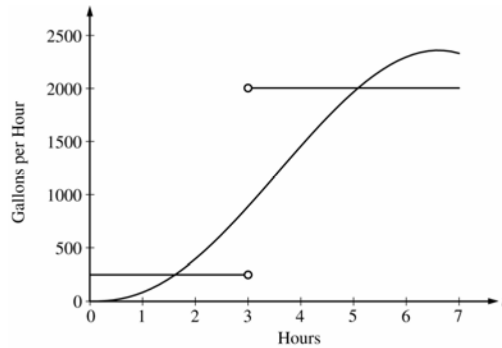
Question 2

The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where t is measured in hours. In this model, rates are given as follows:

(i) The rate at which water enters the tank is $f(t) = 100t^2 \sin\left(\sqrt{t}\right)$ gallons per hour for $0 \le t \le 7$.

(ii) The rate at which water leaves the tank is

$$g(t) = \begin{cases} 250 & \text{for } 0 \le t < 3 \\ 2000 & \text{for } 3 < t \le 7 \end{cases} \text{ gallons per hour.}$$



The graphs of f and g, which intersect at $t = 1.617$ and $t = 5.076$, are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

(a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

(b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for each answer.

(c) For $0 \le t \le 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

(a) $\int_0^7 f(t)\,dt \approx 8264$ gallons

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) The amount of water in the tank is decreasing on the intervals $0 \le t \le 1.617$ and $3 \le t \le 5.076$ because $f(t) < g(t)$ for $0 \le t < 1.617$ and $3 < t < 5.076$.

$2: \begin{cases} 1: \text{intervals} \\ 1: \text{reason} \end{cases}$

(c) Since $f(t) - g(t)$ changes sign from positive to negative only at $t = 3$, the candidates for the absolute maximum are at $t = 0$, 3, and 7.

t (hours)	gallons of water
0	5000
3	$5000 + \int_0^3 f(t)\,dt - 250(3) = 5126.591$
7	$5126.591 + \int_3^7 f(t)\,dt - 2000(4) = 4513.807$

The amount of water in the tank is greatest at 3 hours. At that time, the amount of water in the tank, rounded to the nearest gallon, is 5127 gallons.

$5: \begin{cases} 1: \text{identifies } t = 3 \text{ as a candidate} \\ 1: \text{integrand} \\ 1: \text{amount of water at } t = 3 \\ 1: \text{amount of water at } t = 7 \\ 1: \text{conclusion} \end{cases}$

AP® CALCULUS AB
2008 SCORING GUIDELINES (Form B)

Question 2

For time $t \geq 0$ hours, let $r(t) = 120\left(1 - e^{-10t^2}\right)$ represent the speed, in kilometers per hour, at which a car travels along a straight road. The number of liters of gasoline used by the car to travel x kilometers is modeled by $g(x) = 0.05x\left(1 - e^{-x/2}\right)$.

(a) How many kilometers does the car travel during the first 2 hours?

(b) Find the rate of change with respect to time of the number of liters of gasoline used by the car when $t = 2$ hours. Indicate units of measure.

(c) How many liters of gasoline have been used by the car when it reaches a speed of 80 kilometers per hour?

(a) $\int_0^2 r(t)\,dt = 206.370$ kilometers

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\dfrac{dg}{dt} = \dfrac{dg}{dx} \cdot \dfrac{dx}{dt}; \quad \dfrac{dx}{dt} = r(t)$

$\left.\dfrac{dg}{dt}\right|_{t=2} = \left.\dfrac{dg}{dx}\right|_{x=206.370} \cdot r(2)$

$= (0.050)(120) = 6$ liters/hour

$3 : \begin{cases} 2 : \text{uses chain rule} \\ 1 : \text{answer with units} \end{cases}$

(c) Let T be the time at which the car's speed reaches 80 kilometers per hour.

Then, $r(T) = 80$ or $T = 0.331453$ hours.

At time T, the car has gone

$x(T) = \int_0^T r(t)\,dt = 10.794097$ kilometers

and has consumed $g(x(T)) = 0.537$ liters of gasoline.

$4 : \begin{cases} 1 : \text{equation } r(t) = 80 \\ 2 : \text{distance integral} \\ 1 : \text{answer} \end{cases}$

Question 3

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h.$)

(a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

(b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.

(c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

(a) When $r = 100$ cm and $h = 0.5$ cm, $\frac{dV}{dt} = 2000 \text{ cm}^3/\text{min}$ and $\frac{dr}{dt} = 2.5 \text{ cm}/\text{min}$.

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$$

$$2000 = 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.038 \text{ or } 0.039 \text{ cm}/\text{min}$$

$4: \begin{cases} 1: \frac{dV}{dt} = 2000 \text{ and } \frac{dr}{dt} = 2.5 \\ 2: \text{expression for } \frac{dV}{dt} \\ 1: \text{answer} \end{cases}$

(b) $\frac{dV}{dt} = 2000 - R(t)$, so $\frac{dV}{dt} = 0$ when $R(t) = 2000$.

This occurs when $t = 25$ minutes.

Since $\frac{dV}{dt} > 0$ for $0 < t < 25$ and $\frac{dV}{dt} < 0$ for $t > 25$, the oil slick reaches its maximum volume 25 minutes after the device begins working.

$3: \begin{cases} 1: R(t) = 2000 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(c) The volume of oil, in cm^3, in the slick at time $t = 25$ minutes is given by $60{,}000 + \int_0^{25} (2000 - R(t))\, dt$.

$2: \begin{cases} 1: \text{limits and initial condition} \\ 1: \text{integrand} \end{cases}$

1989 AB6

Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well; that is, $\frac{dy}{dt} = ky$, where y is the amount of oil left in the well at any time t. Initially there were $1{,}000{,}000$ gallons of oil in the well, and 6 years later there were $500{,}000$ gallons remaining. It will no longer be profitable to pump oil when there are fewer than 50,000 gallons remaining.

(a) Write an equation for y, the amount of oil remaining in the well at any time t.

(b) At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining?

(c) In order not to lose money, at what time t should oil no longer be pumped from the well?

1989 AB6
Solution

(a) $\dfrac{dy}{dt} = ky$ or $\begin{cases} \dfrac{dy}{y} = k\,dt \\ \ln|y| = kt + C_1 \\ y = e^{kt+C_1} \end{cases}$

$y = Ce^{kt}$

$$t = 0 \Rightarrow C = 10^6,\ C_1 = \ln 10^6$$
$$\therefore y = 10^6 e^{kt}$$
$$t = 6 \Rightarrow \frac{1}{2} = e^{6k}$$
$$\therefore k = -\frac{\ln 2}{6}$$
$$y = 10^6 e^{\frac{-t}{6}\ln 2} = 10^6 \cdot 2^{\frac{-t}{6}}$$

(b) $\dfrac{dy}{dt} = ky = -\dfrac{\ln 2}{6} \cdot 6 \cdot 10^5$

$= -10^5 \ln 2$

Decreasing at $10^5 \ln 2$ gal/year

(c) $5 \cdot 10^4 = 10^6 e^{kt}$

$$\therefore kt = -\ln 20$$
$$\therefore t = \frac{-\ln 20}{\frac{-\ln 2}{6}}$$
$$= 6\frac{\ln 20}{\ln 2} = 6\log_2 20$$

$6\dfrac{\ln 20}{\ln 2}$ years after starting

5. The temperature outside a house during a 24-hour period is given by

$$F(t) = 80 - 10\cos\left(\frac{\pi t}{12}\right),\ 0 \le t \le 24,$$

where $F(t)$ is measured in degrees Fahrenheit and t is measured in hours.

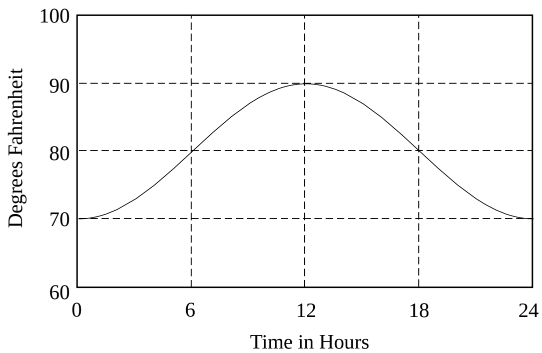
(a) Sketch the graph of F on the grid below.

(b) Find the average temperature, to the nearest degree Fahrenheit, between $t = 6$ and $t = 14$.

(c) An air conditioner cooled the house whenever the outside temperature was at or above 78 degrees Fahrenheit. For what values of t was the air conditioner cooling the house?

(d) The cost of cooling the house accumulates at the rate of \$0.05 per hour for each degree the outside temperature exceeds 78 degrees Fahrenheit. What was the total cost, to the nearest cent, to cool the house for this 24–hour period?

(a)



1: bell–shaped graph
minimum 70 at $t = 0$, $t = 24$ only
maximum 90 at $t = 12$ only

(b) $$\text{Avg.} = \frac{1}{14-6}\int_6^{14}\left[80 - 10\cos\left(\frac{\pi t}{12}\right)\right]dt$$

$$= \frac{1}{8}(697.2957795)$$

$$= 87.162 \quad \text{or} \quad 87.161$$

$$\approx 87^\circ \text{ F}$$

3 $\begin{cases} 2: \text{ integral} \\ \quad 1: \text{ limits and } 1/(14-6) \\ \quad 1: \text{ integrand} \\ 1: \text{ answer} \\ \quad 0/1 \text{ if integral not of the form} \\ \quad \frac{1}{b-a}\int_a^b F(t)\,dt \end{cases}$

(c) $$\left[80 - 10\cos\left(\frac{\pi t}{12}\right)\right] - 78 \ge 0$$

$$2 - 10\cos\left(\frac{\pi t}{12}\right) \ge 0$$

$$\left.\begin{matrix} 5.230 \\ \text{or} \\ 5.231 \end{matrix}\right\} \le t \le \left\{\begin{matrix} 18.769 \\ \text{or} \\ 18.770 \end{matrix}\right.$$

2 $\begin{cases} 1: \text{ inequality or equation} \\ 1: \text{ solutions with interval} \end{cases}$

(d) $$C = 0.05\int_{5.231 \text{ or } 5.230}^{18.770 \text{ or } 18.769}\left(\left[80 - 10\cos\left(\frac{\pi t}{12}\right)\right] - 78\right)dt$$

$$= 0.05(101.92741) = 5.096 \approx \$5.10$$

3 $\begin{cases} 2: \text{ integral} \\ \quad 1: \text{ limits and } 0.05 \\ \quad 1: \text{ integrand} \\ 1: \text{ answer} \\ \quad 0/1 \text{ if integral not of the form} \\ \quad k\int_a^b (F(t) - 78)\,dt \end{cases}$

Water is pumped into an underground tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, for $0 \le t \le 120$ minutes. At time $t = 0$, the tank contains 30 gallons of water.

(a) How many gallons of water leak out of the tank from time $t = 0$ to $t = 3$ minutes?

(b) How many gallons of water are in the tank at time $t = 3$ minutes?

(c) Write an expression for $A(t)$, the total number of gallons of water in the tank at time t.

(d) At what time t, for $0 \le t \le 120$, is the amount of water in the tank a maximum? Justify your answer.

(a) Method 1: $\int_0^3 \sqrt{t+1}\,dt = \frac{2}{3}(t+1)^{3/2} \Big|_0^3 = \frac{14}{3}$

– or –

Method 2: $L(t)$ = gallons leaked in first t minutes

$\frac{dL}{dt} = \sqrt{t+1}$; $\quad L(t) = \frac{2}{3}(t+1)^{3/2} + C$

$L(0) = 0$; $\quad C = -\frac{2}{3}$

$L(t) = \frac{2}{3}(t+1)^{3/2} - \frac{2}{3}$; $\quad L(3) = \frac{14}{3}$

Scoring:

Method 1:
3 { 2 : definite integral (1 : limits, 1 : integrand); 1 : answer }

– or –

Method 2:
3 { 1 : antiderivative with C; 1 : solves for C using $L(0) = 0$; 1 : answer }

(b) $30 + 8 \cdot 3 - \frac{14}{3} = \frac{148}{3}$

Scoring: 1 : answer

(c) Method 1:

$A(t) = 30 + \int_0^t \left(8 - \sqrt{x+1}\right) dx$

$= 30 + 8t - \int_0^t \sqrt{x+1}\,dx$

– or –

Method 2:

$\frac{dA}{dt} = 8 - \sqrt{t+1}$

$A(t) = 8t - \frac{2}{3}(t+1)^{3/2} + C$

$30 = 8(0) - \frac{2}{3}(0+1)^{3/2} + C$; $\quad C = \frac{92}{3}$

$A(t) = 8t - \frac{2}{3}(t+1)^{3/2} + \frac{92}{3}$

Scoring:

Method 1:
2 { 1 : $30 + 8t$; 1 : $-\int_0^t \sqrt{x+1}\,dx$ }

– or –

Method 2:
2 { 1 : antiderivative with C; 1 : answer }

(d) $A'(t) = 8 - \sqrt{t+1} = 0$ when $t = 63$

$A'(t)$ is positive for $0 < t < 63$ and negative for $63 < t < 120$. Therefore there is a maximum at $t = 63$.

Scoring:
3 { 1 : sets $A'(t) = 0$; 1 : solves for t; 1 : justification }

Question 2

The number of gallons, $P(t)$, of a pollutant in a lake changes at the rate $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$ gallons per day, where t is measured in days. There are 50 gallons of the pollutant in the lake at time $t = 0$. The lake is considered to be safe when it contains 40 gallons or less of pollutant.

(a) Is the amount of pollutant increasing at time $t = 9$? Why or why not?

(b) For what value of t will the number of gallons of pollutant be at its minimum? Justify your answer.

(c) Is the lake safe when the number of gallons of pollutant is at its minimum? Justify your answer.

(d) An investigator uses the tangent line approximation to $P(t)$ at $t = 0$ as a model for the amount of pollutant in the lake. At what time t does this model predict that the lake becomes safe?

(a) $P'(9) = 1 - 3e^{-0.6} = -0.646 < 0$ so the amount is not increasing at this time.	1 : answer with reason
(b) $P'(t) = 1 - 3e^{-0.2\sqrt{t}} = 0$ $t = (5\ln 3)^2 = 30.174$ $P'(t)$ is negative for $0 < t < (5\ln 3)^2$ and positive for $t > (5\ln 3)^2$. Therefore there is a minimum at $t = (5\ln 3)^2$.	3 { 1 : sets $P'(t) = 0$ 1 : solves for t 1 : justification
(c) $P(30.174) = 50 + \int_0^{30.174} \left(1 - 3e^{-0.2\sqrt{t}}\right) dt$ $= 35.104 < 40$, so the lake is safe.	3 { 1 : integrand 1 : limits 1 : conclusion with reason based on integral of $P'(t)$
(d) $P'(0) = 1 - 3 = -2$. The lake will become safe when the amount decreases by 10. A linear model predicts this will happen when $t = 5$.	2 { 1 : slope of tangent line 1 : answer

3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24–hour period.

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

(a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t)\,dt$. Using correct units, explain the meaning of your answer in terms of water flow.

(b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

(c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$. Use $Q(t)$ to approximate the average rate of water flow during the 24–hour time period. Indicate units of measure.

(a) $\int_0^{24} R(t)\,dt \approx 6[R(3) + R(9) + R(15) + R(21)]$

$= 6[10.4 + 11.2 + 11.3 + 10.2]$

$= 258.6$ gallons

This is an approximation to the total flow in gallons of water from the pipe in the 24–hour period.

3 { 1: $R(3) + R(9) + R(15) + R(21)$; 1: answer; 1: explanation }

(b) Yes;

Since $R(0) = R(24) = 9.6$, the Mean Value Theorem guarantees that there is a t, $0 < t < 24$, such that $R'(t) = 0$.

2 { 1: answer; 1: MVT or equivalent }

(c) Average rate of flow

$\approx$ average value of $Q(t)$

$$= \frac{1}{24}\int_0^{24} \frac{1}{79}(768 + 23t - t^2)\,dt$$

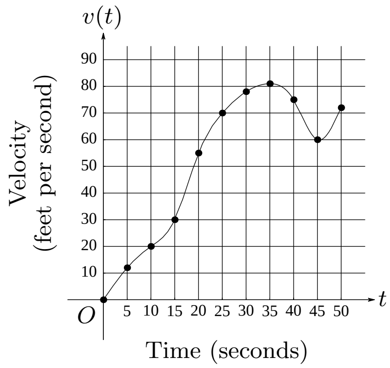
$= 10.785$ gal/hr or 10.784 gal/hr

3 { 1: limits and average value constant; 1: $Q(t)$ as integrand; 1: answer }

(units) Gallons in part (a) and gallons/hr in part (c), or equivalent.

1: units

1998 Calculus AB Scoring Guidelines



t (seconds)	$v(t)$ (feet per second)
0	0
5	12
10	20
15	30
20	55
25	70
30	78
35	81
40	75
45	60
50	72

3. The graph of the velocity $v(t)$, in ft/sec, of a car traveling on a straight road, for $0 \le t \le 50$, is shown above. A table of values for $v(t)$, at 5 second intervals of time t, is shown to the right of the graph.
   (a) During what intervals of time is the acceleration of the car positive? Give a reason for your answer.
   (b) Find the average acceleration of the car, in ft/sec^2, over the interval $0 \le t \le 50$.
   (c) Find one approximation for the acceleration of the car, in ft/sec^2, at $t = 40$. Show the computations you used to arrive at your answer.
   (d) Approximate $\int_0^{50} v(t)\,dt$ with a Riemann sum, using the midpoints of five subintervals of equal length. Using correct units, explain the meaning of this integral.

(a) Acceleration is positive on $(0, 35)$ and $(45, 50)$ because the velocity $v(t)$ is increasing on $[0, 35]$ and $[45, 50]$

3 { 1: $(0, 35)$; 1: $(45, 50)$; 1: reason }

Note: ignore inclusion of endpoints

(b) Avg. Acc. $= \dfrac{v(50) - v(0)}{50 - 0} = \dfrac{72 - 0}{50} = \dfrac{72}{50}$

or 1.44 ft/sec^2

1: answer

(c) Difference quotient; e.g.

$$\frac{v(45) - v(40)}{5} = \frac{60 - 75}{5} = -3 \text{ ft/sec}^2 \quad \text{or}$$

$$\frac{v(40) - v(35)}{5} = \frac{75 - 81}{5} = -\frac{6}{5} \text{ ft/sec}^2 \quad \text{or}$$

$$\frac{v(45) - v(35)}{10} = \frac{60 - 81}{10} = -\frac{21}{10} \text{ ft/sec}^2$$

–or–

Slope of tangent line, e.g.

through $(35, 90)$ and $(40, 75)$: $\dfrac{90 - 75}{35 - 40} = -3$ ft/sec^2

2 { 1: method; 1: answer }

Note: 0/2 if first point not earned

(d) $\int_0^{50} v(t)\,dt$

$\approx 10[v(5) + v(15) + v(25) + v(35) + v(45)]$

$= 10(12 + 30 + 70 + 81 + 60)$

$= 2530$ feet

This integral is the total distance traveled in feet over the time 0 to 50 seconds.

3 { 1: midpoint Riemann sum; 1: answer; 1: meaning of integral }

Question 2

The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

t (days)	$W(t)$ (°C)
0	20
3	31
6	28
9	24
12	22
15	21

(a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

(c) A student proposes the function P, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

(d) Use the function P defined in part (c) to find the average value, in degrees Celsius, of $P(t)$ over the time interval $0 \le t \le 15$ days.

(a) Difference quotient; e.g.

$$W'(12) \approx \frac{W(15) - W(12)}{15 - 12} = -\frac{1}{3} \text{ °C/day or}$$

$$W'(12) \approx \frac{W(12) - W(9)}{12 - 9} = -\frac{2}{3} \text{ °C/day or}$$

$$W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = -\frac{1}{2} \text{ °C/day}$$

2 : { 1 : difference quotient; 1 : answer (with units) }

(b) $\frac{3}{2}(20 + 2(31) + 2(28) + 2(24) + 2(22) + 21) = 376.5$

Average temperature $\approx \frac{1}{15}(376.5) = 25.1$ °C

2 : { 1 : trapezoidal method; 1 : answer }

(c) $P'(12) = 10e^{-t/3} - \frac{10}{3}te^{-t/3}\Big|_{t=12}$

$= -30e^{-4} = -0.549$ °C/day

This means that the temperature is decreasing at the rate of 0.549 °C/day when $t = 12$ days.

2 : { 1 : $P'(12)$ (with or without units); 1 : interpretation }

(d) $\frac{1}{15}\int_0^{15}\left(20 + 10te^{-t/3}\right)dt = 25.757$ °C

3 : { 1 : integrand; 1 : limits and average value constant; 1 : answer }

Question 3

Distance x (mm)	0	60	120	180	240	300	360
Diameter $B(x)$ (mm)	24	30	28	30	26	24	26

A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points along the length of the blood vessel, where x represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

(a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.

(b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.

(c) Using correct units, explain the meaning of $\pi \int_{125}^{275} \left(\frac{B(x)}{2} \right)^2 dx$ in terms of the blood vessel.

(d) Explain why there must be at least one value x, for $0 < x < 360$, such that $B''(x) = 0$.

(a) $\frac{1}{360} \int_0^{360} \frac{B(x)}{2} \, dx$

2 : { 1 : limits and constant; 1 : integrand }

(b) $\frac{1}{360} \left[120 \left(\frac{B(60)}{2} + \frac{B(180)}{2} + \frac{B(300)}{2} \right) \right] =$

$\frac{1}{360} [60(30 + 30 + 24)] = 14$

2 : { 1 : $B(60) + B(180) + B(300)$; 1 : answer }

(c) $\frac{B(x)}{2}$ is the radius, so $\pi \left(\frac{B(x)}{2} \right)^2$ is the area of the cross section at x. The expression is the volume in mm^3 of the blood vessel between 125 mm and 275 mm from the end of the vessel.

2 : { 1 : volume in mm^3; 1 : between $x = 125$ and $x = 275$ }

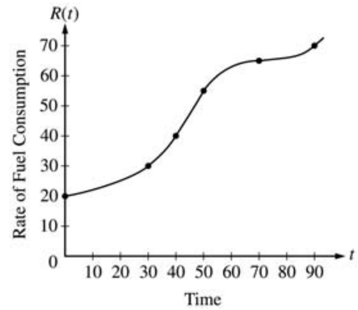
(d) By the MVT, $B'(c_1) = 0$ for some c_1 in (60, 180) and $B'(c_2) = 0$ for some c_2 in (240, 360). The MVT applied to $B'(x)$ shows that $B''(x) = 0$ for some x in the interval (c_1, c_2).

3 : { 2 : explains why there are two values of x where $B'(x)$ has the same value; 1 : explains why that means $B''(x) = 0$ for $0 < x < 360$ }

Note: max 1/3 if only explains why $B'(x) = 0$ at some x in (0, 360).

Question 3

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

(a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

(b) The rate of fuel consumption is increasing fastest at time $t = 45$ minutes. What is the value of $R''(45)$? Explain your reasoning.

(c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$? Explain your reasoning.

(d) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

(a) $R'(45) \approx \dfrac{R(50) - R(40)}{50 - 40} = \dfrac{55 - 40}{10}$
$= 1.5 \text{ gal/min}^2$

2 : { 1 : a difference quotient using numbers from table and interval that contains 45; 1 : 1.5 gal/min^2 }

(b) $R''(45) = 0$ since $R'(t)$ has a maximum at $t = 45$.

2 : { 1 : $R''(45) = 0$; 1 : reason }

(c) $\int_0^{90} R(t)\,dt \approx (30)(20) + (10)(30) + (10)(40)$
$+(20)(55) + (20)(65) = 3700$

Yes, this approximation is less because the graph of R is increasing on the interval.

2 : { 1 : value of left Riemann sum; 1 : "less" with reason }

(d) $\int_0^{b} R(t)\,dt$ is the total amount of fuel in gallons consumed for the first b minutes.

$\frac{1}{b}\int_0^{b} R(t)\,dt$ is the average value of the rate of fuel consumption in gallons/min during the first b minutes.

3 : { 2 : meanings
1 : meaning of $\int_0^{b} R(t)\,dt$
1 : meaning of $\frac{1}{b}\int_0^{b} R(t)\,dt$
$< -1 >$ if no reference to time b
1 : units in both answers }

Question 3

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

t (min)	0	5	10	15	20	25	30	35	40
$v(t)$ (mpm)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

(c) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicates units of measure.

(d) According to the model f, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

(a) Midpoint Riemann sum is

$10 \cdot [v(5) + v(15) + v(25) + v(35)]$

$= 10 \cdot [9.2 + 7.0 + 2.4 + 4.3] = 229$

The integral gives the total distance in miles that the plane flies during the 40 minutes.

3 : { 1 : $v(5) + v(15) + v(25) + v(35)$; 1 : answer; 1 : meaning with units }

(b) By the Mean Value Theorem, $v'(t) = 0$ somewhere in the interval $(0, 15)$ and somewhere in the interval $(25, 30)$. Therefore the acceleration will equal 0 for at least two values of t.

2 : { 1 : two instances; 1 : justification }

(c) $f'(23) = -0.407$ or -0.408 miles per minute^2

1 : answer with units

(d) Average velocity $= \frac{1}{40}\int_0^{40} f(t)\,dt$

$= 5.916$ miles per minute

3 : { 1 : limits; 1 : integrand; 1 : answer }

Question 3

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

(a) $\dfrac{T(8)-T(6)}{8-6} = \dfrac{55-62}{2} = -\dfrac{7}{2}\,°\text{C/cm}$

1 : answer

(b) $\dfrac{1}{8}\int_0^8 T(x)\,dx$

Trapezoidal approximation for $\int_0^8 T(x)\,dx$:

$$A = \frac{100+93}{2}\cdot 1 + \frac{93+70}{2}\cdot 4 + \frac{70+62}{2}\cdot 1 + \frac{62+55}{2}\cdot 2$$

Average temperature $\approx \dfrac{1}{8}A = 75.6875°\text{C}$

3 : { 1 : $\frac{1}{8}\int_0^8 T(x)\,dx$; 1 : trapezoidal sum; 1 : answer }

(c) $\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45°\text{C}$

The temperature drops 45°C from the heated end of the wire to the other end of the wire.

2 : { 1 : value; 1 : meaning }

(d) Average rate of change of temperature on $[1, 5]$ is $\dfrac{70-93}{5-1} = -5.75$.

Average rate of change of temperature on $[5, 6]$ is $\dfrac{62-70}{6-5} = -8$.

No. By the MVT, $T'(c_1) = -5.75$ for some c_1 in the interval $(1, 5)$ and $T'(c_2) = -8$ for some c_2 in the interval $(5, 6)$. It follows that T' must decrease somewhere in the interval (c_1, c_2). Therefore T'' is not positive for every x in $[0, 8]$.

2 : { 1 : two slopes of secant lines; 1 : answer with explanation }

Units of °C/cm in (a), and °C in (b) and (c)

1 : units in (a), (b), and (c)

Question 6

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	–20	–30	–20	–14	–10	0	10
$a(t)$ (ft/sec^2)	1	5	2	1	2	4	2

A car travels on a straight track. During the time interval $0 \le t \le 60$ seconds, the car's velocity v, measured in feet per second, and acceleration a, measured in feet per second per second, are continuous functions. The table above shows selected values of these functions.

(a) Using appropriate units, explain the meaning of $\int_{30}^{60} |v(t)|\, dt$ in terms of the car's motion. Approximate $\int_{30}^{60} |v(t)|\, dt$ using a trapezoidal approximation with the three subintervals determined by the table.

(b) Using appropriate units, explain the meaning of $\int_{0}^{30} a(t)\, dt$ in terms of the car's motion. Find the exact value of $\int_{0}^{30} a(t)\, dt$.

(c) For $0 < t < 60$, must there be a time t when $v(t) = -5$? Justify your answer.

(d) For $0 < t < 60$, must there be a time t when $a(t) = 0$? Justify your answer.

(a) $\int_{30}^{60} \lvert v(t)\rvert\, dt$ is the distance in feet that the car travels from $t = 30$ sec to $t = 60$ sec. Trapezoidal approximation for $\int_{30}^{60} \lvert v(t)\rvert\, dt$: $A = \frac{1}{2}(14 + 10)5 + \frac{1}{2}(10)(15) + \frac{1}{2}(10)(10) = 185$ ft	2 : { 1 : explanation 1 : value
(b) $\int_{0}^{30} a(t)\, dt$ is the car's change in velocity in ft/sec from $t = 0$ sec to $t = 30$ sec. $\int_{0}^{30} a(t)\, dt = \int_{0}^{30} v'(t)\, dt = v(30) - v(0)$ $= -14 - (-20) = 6$ ft/sec	2 : { 1 : explanation 1 : value
(c) Yes. Since $v(35) = -10 < -5 < 0 = v(50)$, the IVT guarantees a t in $(35, 50)$ so that $v(t) = -5$.	2 : { 1 : $v(35) < -5 < v(50)$ 1 : Yes; refers to IVT or hypotheses
(d) Yes. Since $v(0) = v(25)$, the MVT guarantees a t in $(0, 25)$ so that $a(t) = v'(t) = 0$.	2 : { 1 : $v(0) = v(25)$ 1 : Yes; refers to MVT or hypotheses
Units of ft in (a) and ft/sec in (b)	1 : units in (a) and (b)

Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

(a) Average acceleration of rocket A is

$$\frac{v(80) - v(0)}{80 - 0} = \frac{49 - 5}{80} = \frac{11}{20}\ \text{ft}/\text{sec}^2$$

1 : answer

(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds.

A midpoint Riemann sum is

$$20[v(20) + v(40) + v(60)]$$
$$= 20[22 + 35 + 44] = 2020 \text{ ft}$$

3 : { 1 : explanation; 1 : uses $v(20)$, $v(40)$, $v(60)$; 1 : value }

(c) Let $v_B(t)$ be the velocity of rocket B at time t.

$$v_B(t) = \int \frac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$$
$$2 = v_B(0) = 6 + C$$
$$v_B(t) = 6\sqrt{t+1} - 4$$
$$v_B(80) = 50 > 49 = v(80)$$

Rocket B is traveling faster at time $t = 80$ seconds.

4 : { 1 : $6\sqrt{t+1}$; 1 : constant of integration; 1 : uses initial condition; 1 : finds $v_B(80)$, compares to $v(80)$, and draws a conclusion }

Units of ft/sec^2 in (a) and ft in (b)

1 : units in (a) and (b)

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Question 5

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

(a) $r(5.4) \approx r(5) + r'(5)\Delta t = 30 + 2(0.4) = 30.8$ ft Since the graph of r is concave down on the interval $5 < t < 5.4$, this estimate is greater than $r(5.4)$.	2 : { 1 : estimate 1 : conclusion with reason
(b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$ $\left.\frac{dV}{dt}\right\|_{t=5} = 4\pi(30)^2\, 2 = 7200\pi \text{ ft}^3/\text{min}$	3 : { 2 : $\frac{dV}{dt}$ 1 : answer
(c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$ $= 19.3$ ft $\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.	2 : { 1 : approximation 1 : explanation
(d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$.	1 : conclusion with reason
Units of ft^3/min in part (b) and ft in part (c)	1 : units in (b) and (c)

Question 3

Distance from the river's edge (feet)	0	8	14	22	24
Depth of the water (feet)	0	7	8	2	0

A scientist measures the depth of the Doe River at Picnic Point. The river is 24 feet wide at this location. The measurements are taken in a straight line perpendicular to the edge of the river. The data are shown in the table above. The velocity of the water at Picnic Point, in feet per minute, is modeled by $v(t) = 16 + 2\sin\left(\sqrt{t+10}\right)$ for $0 \le t \le 120$ minutes.

(a) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the area of the cross section of the river at Picnic Point, in square feet. Show the computations that lead to your answer.

(b) The volumetric flow at a location along the river is the product of the cross-sectional area and the velocity of the water at that location. Use your approximation from part (a) to estimate the average value of the volumetric flow at Picnic Point, in cubic feet per minute, from $t = 0$ to $t = 120$ minutes.

(c) The scientist proposes the function f, given by $f(x) = 8\sin\left(\frac{\pi x}{24}\right)$, as a model for the depth of the water, in feet, at Picnic Point x feet from the river's edge. Find the area of the cross section of the river at Picnic Point based on this model.

(d) Recall that the volumetric flow is the product of the cross-sectional area and the velocity of the water at a location. To prevent flooding, water must be diverted if the average value of the volumetric flow at Picnic Point exceeds 2100 cubic feet per minute for a 20-minute period. Using your answer from part (c), find the average value of the volumetric flow during the time interval $40 \le t \le 60$ minutes. Does this value indicate that the water must be diverted?

Solution	Scoring
(a) $\frac{(0+7)}{2}\cdot 8 + \frac{(7+8)}{2}\cdot 6 + \frac{(8+2)}{2}\cdot 8 + \frac{(2+0)}{2}\cdot 2$ $= 115 \text{ ft}^2$	1 : trapezoidal approximation
(b) $\frac{1}{120}\int_0^{120} 115v(t)\,dt$ $= 1807.169$ or 1807.170 ft^3/min	3 : { 1 : limits and average value constant; 1 : integrand; 1 : answer }
(c) $\int_0^{24} 8\sin\left(\frac{\pi x}{24}\right)dx = 122.230$ or 122.231 ft^2	2 : { 1 : integra1; 1 : answer }
(d) Let C be the cross-sectional area approximation from part (c). The average volumetric flow is $\frac{1}{20}\int_{40}^{60} C\cdot v(t)\,dt = 2181.912$ or 2181.913 ft^3/min. Yes, water must be diverted since the average volumetric flow for this 20-minute period exceeds 2100 ft^3/min.	3 : { 1 : volumetric flow integral; 1 : average volumetric flow; 1 : answer with reason }

Question 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

(a) $L'(5.5) \approx \dfrac{L(7) - L(4)}{7 - 4} = \dfrac{150 - 126}{3} = 8$ people per hour

2 : { 1 : estimate; 1 : units }

(b) The average number of people waiting in line during the first 4 hours is approximately

$$\frac{1}{4}\left(\frac{L(0) + L(1)}{2}(1 - 0) + \frac{L(1) + L(3)}{2}(3 - 1) + \frac{L(3) + L(4)}{2}(4 - 3)\right)$$

$= 155.25$ people

2 : { 1 : trapezoidal sum; 1 : answer }

(c) L is differentiable on $[0, 9]$ so the Mean Value Theorem implies $L'(t) > 0$ for some t in $(1, 3)$ and some t in $(4, 7)$. Similarly, $L'(t) < 0$ for some t in $(3, 4)$ and some t in $(7, 8)$. Then, since L' is continuous on $[0, 9]$, the Intermediate Value Theorem implies that $L'(t) = 0$ for at least three values of t in $[0, 9]$.

3 : { 1 : considers change in sign of L'; 1 : analysis; 1 : conclusion }

OR

The continuity of L on $[1, 4]$ implies that L attains a maximum value there. Since $L(3) > L(1)$ and $L(3) > L(4)$, this maximum occurs on $(1, 4)$. Similarly, L attains a minimum on $(3, 7)$ and a maximum on $(4, 8)$. L is differentiable, so $L'(t) = 0$ at each relative extreme point on $(0, 9)$. Therefore $L'(t) = 0$ for at least three values of t in $[0, 9]$.

OR

3 : { 1 : considers relative extrema of L on $(0, 9)$; 1 : analysis; 1 : conclusion }

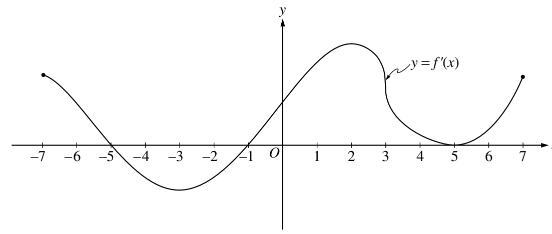
[*Note: There is a function L that satisfies the given conditions with $L'(t) = 0$ for exactly three values of t.*]

(d) $\int_0^3 r(t)\, dt = 972.784$

There were approximately 973 tickets sold by 3 P.M.

2 : { 1 : integrand; 1 : limits and answer }

The figure above shows the graph of f', the derivative of the function f, for $-7 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$, and $x = 5$, and a vertical tangent line at $x = 3$.



(a) Find all values of x, for $-7 < x < 7$, at which f attains a relative minimum. Justify your answer.

(b) Find all values of x, for $-7 < x < 7$, at which f attains a relative maximum. Justify your answer.

(c) Find all values of x, for $-7 < x < 7$, at which $f''(x) < 0$.

(d) At what value of x, for $-7 \le x \le 7$, does f attain its absolute maximum? Justify your answer.

(a) $x = -1$

$f'(x)$ changes from negative to positive at $x = -1$

$2 \begin{cases} 1: \text{ answer} \\ 1: \text{ justification} \end{cases}$

(b) $x = -5$

$f'(x)$ changes from positive to negative at $x = -5$

$2 \begin{cases} 1: \text{ answer} \\ 1: \text{ justification} \end{cases}$

(c) $f''(x)$ exists and f' is decreasing on the intervals $(-7,-3)$, $(2,3)$, and $(3,5)$

$2 \begin{cases} 1: \ (-7,-3) \\ 1: \ (2,3) \cup (3,5) \end{cases}$

(d) $x = 7$

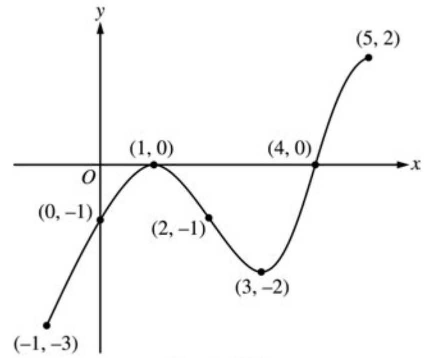
The absolute maximum must occur at $x = -5$ or at an endpoint.

$f(-5) > f(-7)$ because f is increasing on $(-7,-5)$

The graph of f' shows that the magnitude of the negative change in f from $x = -5$ to $x = -1$ is smaller than the positive change in f from $x = -1$ to $x = 7$. Therefore the net change in f is positive from $x = -5$ to $x = 7$, and $f(7) > f(-5)$. So $f(7)$ is the absolute maximum.

3 {
- 1 : answer
- 1 : identifies $x = -5$ and $x = 7$ as candidates
  — or —
  indicates that the graph of f increases, decreases, then increases
- 1 : justifies $f(7) > f(-5)$

Question 4



Graph of f'

The figure above shows the graph of f', the derivative of the function f, on the closed interval $-1 \le x \le 5$. The graph of f' has horizontal tangent lines at $x = 1$ and $x = 3$. The function f is twice differentiable with $f(2) = 6$.

(a) Find the x-coordinate of each of the points of inflection of the graph of f. Give a reason for your answer.

(b) At what value of x does f attain its absolute minimum value on the closed interval $-1 \le x \le 5$? At what value of x does f attain its absolute maximum value on the closed interval $-1 \le x \le 5$? Show the analysis that leads to your answers.

(c) Let g be the function defined by $g(x) = xf(x)$. Find an equation for the line tangent to the graph of g at $x = 2$.

(a) $x = 1$ and $x = 3$ because the graph of f' changes from increasing to decreasing at $x = 1$, and changes from decreasing to increasing at $x = 3$.

$2 : \begin{cases} 1 : x = 1,\ x = 3 \\ 1 : \text{reason} \end{cases}$

(b) The function f decreases from $x = -1$ to $x = 4$, then increases from $x = 4$ to $x = 5$.
Therefore, the absolute minimum value for f is at $x = 4$.
The absolute maximum value must occur at $x = -1$ or at $x = 5$.

$$f(5) - f(-1) = \int_{-1}^{5} f'(t)\,dt < 0$$

Since $f(5) < f(-1)$, the absolute maximum value occurs at $x = -1$.

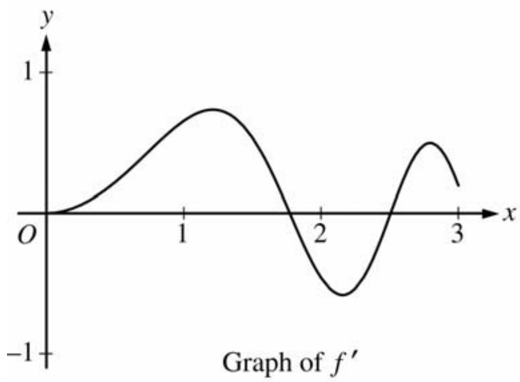
$4 : \begin{cases} 1 : \text{indicates } f \text{ decreases then increases} \\ 1 : \text{eliminates } x = 5 \text{ for maximum} \\ 1 : \text{absolute minimum at } x = 4 \\ 1 : \text{absolute maximum at } x = -1 \end{cases}$

(c) $g'(x) = f(x) + xf'(x)$
$g'(2) = f(2) + 2f'(2) = 6 + 2(-1) = 4$
$g(2) = 2f(2) = 12$

Tangent line is $y = 4(x - 2) + 12$

$3 : \begin{cases} 2 : g'(x) \\ 1 : \text{tangent line} \end{cases}$

Question 2



Let f be the function defined for $x \ge 0$ with $f(0) = 5$ and f', the first derivative of f, given by $f'(x) = e^{(-x/4)} \sin\left(x^2\right)$. The graph of $y = f'(x)$ is shown above.

(a) Use the graph of f' to determine whether the graph of f is concave up, concave down, or neither on the interval $1.7 < x < 1.9$. Explain your reasoning.

(b) On the interval $0 \le x \le 3$, find the value of x at which f has an absolute maximum. Justify your answer.

(c) Write an equation for the line tangent to the graph of f at $x = 2$.

(a) On the interval $1.7 < x < 1.9$, f' is decreasing and thus f is concave down on this interval.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{reason} \end{cases}$

(b) $f'(x) = 0$ when $x = 0, \sqrt{\pi}, \sqrt{2\pi}, \sqrt{3\pi}, \ldots$

On $[0, 3]$ f' changes from positive to negative only at $\sqrt{\pi}$. The absolute maximum must occur at $x = \sqrt{\pi}$ or at an endpoint.

$f(0) = 5$

$f\left(\sqrt{\pi}\right) = f(0) + \int_0^{\sqrt{\pi}} f'(x)\,dx = 5.67911$

$f(3) = f(0) + \int_0^3 f'(x)\,dx = 5.57893$

This shows that f has an absolute maximum at $x = \sqrt{\pi}$.

3 :
- 1 : identifies $\sqrt{\pi}$ and 3 as candidates
  - or -
  indicates that the graph of f increases, decreases, then increases
- 1 : justifies $f\left(\sqrt{\pi}\right) > f(3)$
- 1 : answer

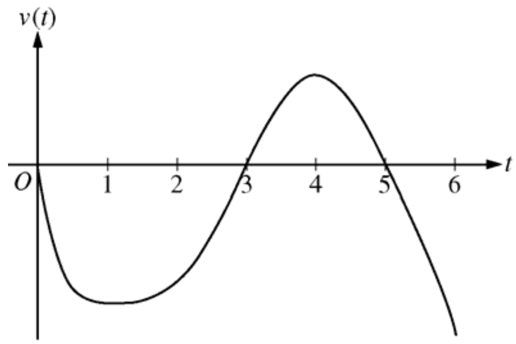
(c) $f(2) = f(0) + \int_0^2 f'(x)\,dx = 5.62342$

$f'(2) = e^{-0.5} \sin(4) = -0.45902$

$y - 5.623 = (-0.459)(x - 2)$

4 :
- 2 : $f(2)$ expression
  - 1 : integral
  - 1 : including $f(0)$ term
- 1 : $f'(2)$
- 1 : equation

Question 4



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

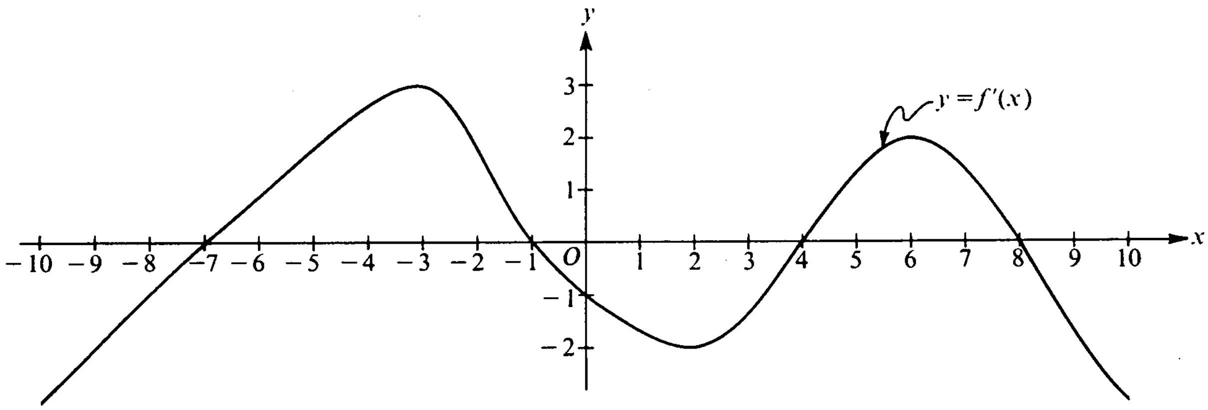
(b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

(a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$. $x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$ $x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$ Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$.	3 : { 1 : identifies $t = 3$ as a candidate 1 : considers $\int_0^6 v(t)\,dt$ 1 : conclusion
(b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$. By the Intermediate Value Theorem, there are three values of t for which the particle is at $x(t) = -8$.	3 : { 1 : positions at $t = 3$, $t = 5$, and $t = 6$ 1 : description of motion 1 : conclusion
(c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.	1 : answer with reason
(d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.	2 : { 1 : answer 1 : justification

1989 AB5



Note: This is the graph of the derivative of f, not the graph of f.

The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-10 \leq x \leq 10$.

(a) For what values of x does the graph of f have a horizontal tangent?

(b) For what values of x in the interval $(-10,10)$ does f have a relative maximum? Justify your answer.

(c) For value of x is the graph of f concave downward?

1989 AB5
Solution

(a) horizontal tangent $\Leftrightarrow f'(x)=0$

$x=-7,-1,4,8$

(b) Relative maxima at $x=-1,\ 8$ because f' changes from positive to negative at these points

(c) f concave downward $\Leftrightarrow f'$ decreasing

$(-3,2)$, $(6,10)$

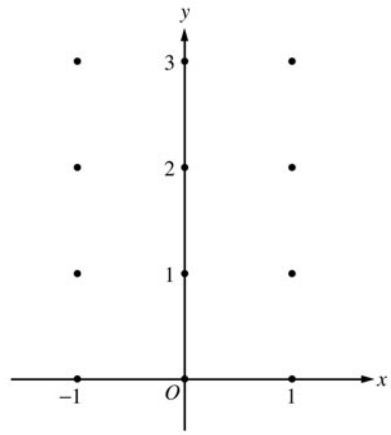
Question 5

Consider the differential equation $\frac{dy}{dx} = x^4(y - 2)$.

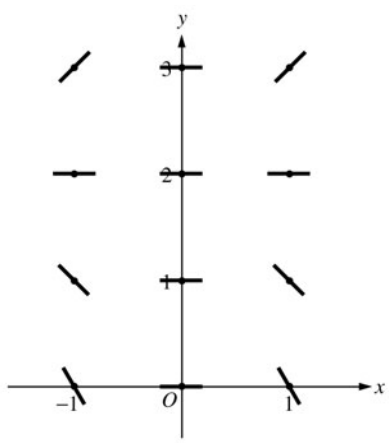
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the test booklet.)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are negative.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 0$.



(a)



2 : $\begin{cases} 1 : \text{zero slope at each point } (x, y) \text{ where } x = 0 \text{ or } y = 2 \\ 1 : \begin{cases} \text{positive slope at each point } (x, y) \text{ where } x \neq 0 \text{ and } y > 2 \\ \text{negative slope at each point } (x, y) \text{ where } x \neq 0 \text{ and } y < 2 \end{cases} \end{cases}$

(b) Slopes are negative at points (x, y) where $x \neq 0$ and $y < 2$.

1 : description

(c) $\frac{1}{y-2}\,dy = x^4\,dx$

$\ln|y - 2| = \frac{1}{5}x^5 + C$

$|y - 2| = e^C e^{\frac{1}{5}x^5}$

$y - 2 = Ke^{\frac{1}{5}x^5}$, $K = \pm e^C$

$-2 = Ke^0 = K$

$y = 2 - 2e^{\frac{1}{5}x^5}$

6 : $\begin{cases} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \\ \quad 0/1 \text{ if } y \text{ is not exponential} \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration
Note: $0/6$ if no separation of variables

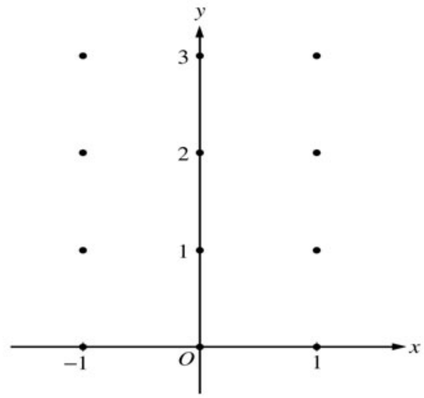
Question 6

Consider the differential equation $\frac{dy}{dx} = x^2(y - 1)$.

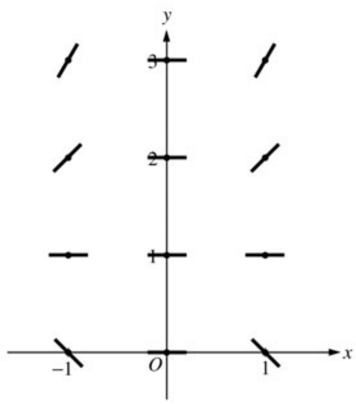
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(**Note: Use the axes provided in the pink test booklet.**)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are positive.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$.



(a)



2 : { 1 : zero slope at each point (x, y) where $x = 0$ or $y = 1$
  1 : { positive slope at each point (x, y) where $x \neq 0$ and $y > 1$
   negative slope at each point (x, y) where $x \neq 0$ and $y < 1$ } }

(b) Slopes are positive at points (x, y) where $x \neq 0$ and $y > 1$.

1 : description

(c)

$$\frac{1}{y-1}dy = x^2 dx$$

$$\ln|y - 1| = \frac{1}{3}x^3 + C$$

$$|y - 1| = e^C e^{\frac{1}{3}x^3}$$

$$y - 1 = Ke^{\frac{1}{3}x^3},\ K = \pm e^C$$

$$2 = Ke^0 = K$$

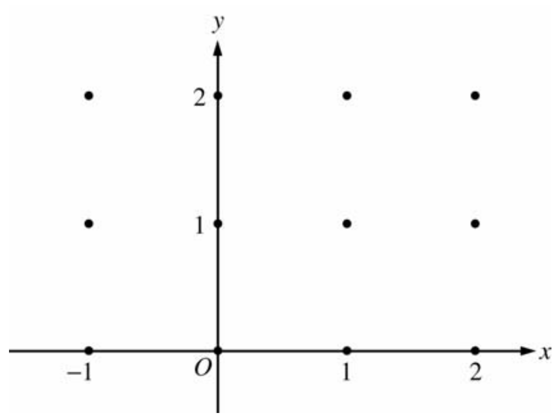
$$y = 1 + 2e^{\frac{1}{3}x^3}$$

6 : { 1 : separates variables
 2 : antiderivatives
 1 : constant of integration
 1 : uses initial condition
 1 : solves for y
  0/1 if y is not exponential }

Note: max 3/6 [1-2-0-0-0] if no constant of integration
Note: 0/6 if no separation of variables

Question 6

Consider the differential equation $\frac{dy}{dx} = \frac{-xy^2}{2}$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 2$.

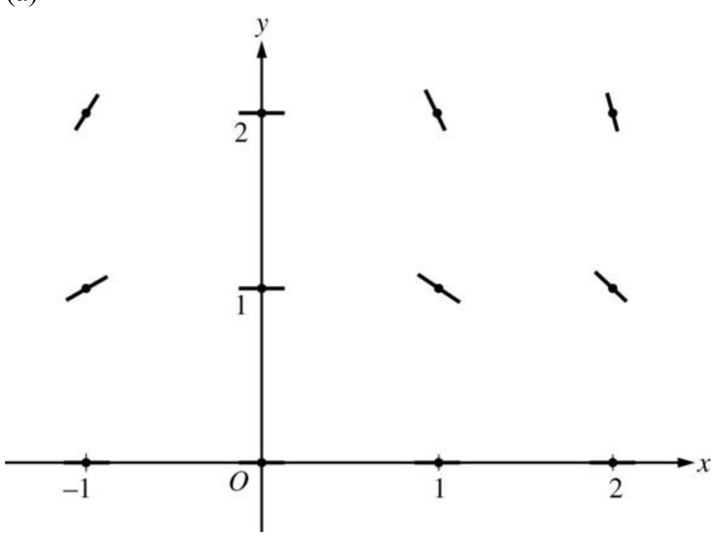


(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the test booklet.)

(b) Write an equation for the line tangent to the graph of f at $x = -1$.

(c) Find the solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \end{cases}$

(b) Slope $= \frac{-(-1)4}{2} = 2$

$y - 2 = 2(x + 1)$

1 : equation

(c) $\frac{1}{y^2}\,dy = -\frac{x}{2}\,dx$

$-\frac{1}{y} = -\frac{x^2}{4} + C$

$-\frac{1}{2} = -\frac{1}{4} + C;\; C = -\frac{1}{4}$

$y = \frac{1}{\frac{x^2}{4} + \frac{1}{4}} = \frac{4}{x^2 + 1}$

$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration
Note: $0/6$ if no separation of variables

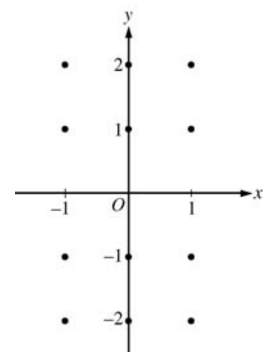
Question 6

Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

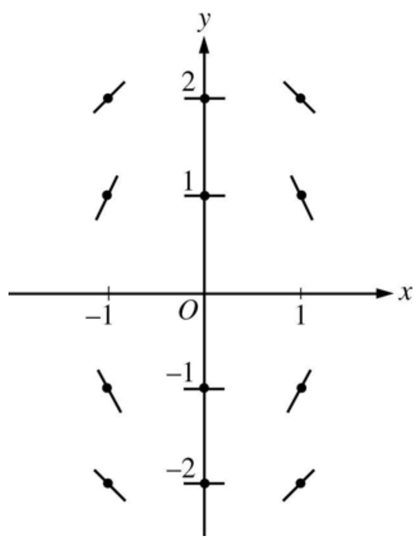
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(**Note: Use the axes provided in the pink test booklet.**)

(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at $(1, -1)$ and use it to approximate $f(1.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.



(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \end{cases}$

(b) The line tangent to f at $(1, -1)$ is $y + 1 = 2(x - 1)$.
Thus, $f(1.1)$ is approximately -0.8.

$2 : \begin{cases} 1 : \text{equation of the tangent line} \\ 1 : \text{approximation for } f(1.1) \end{cases}$

(c) $\frac{dy}{dx} = -\frac{2x}{y}$

$y\,dy = -2x\,dx$

$\frac{y^2}{2} = -x^2 + C$

$\frac{1}{2} = -1 + C;\ C = \frac{3}{2}$

$y^2 = -2x^2 + 3$

Since the particular solution goes through $(1, -1)$, y must be negative.

Thus the particular solution is $y = -\sqrt{3 - 2x^2}$.

$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

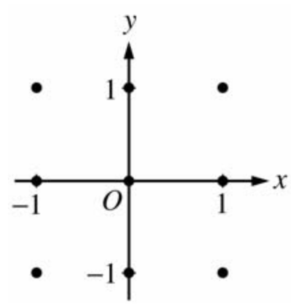
Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Question 5

Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

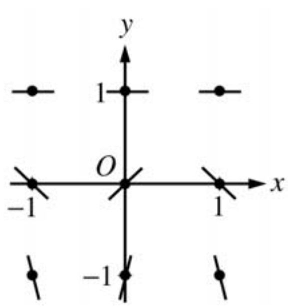
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated. **(Note: Use the axes provided in the exam booklet.)**



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

(b) The line $y = 1$ satisfies the differential equation, so $c = 1$.

$1 : c = 1$

(c)
$$\frac{1}{(y-1)^2}\, dy = \cos(\pi x)\, dx$$
$$-(y-1)^{-1} = \frac{1}{\pi}\sin(\pi x) + C$$
$$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + C$$
$$1 = \frac{1}{\pi}\sin(\pi) + C = C$$
$$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + 1$$
$$\frac{\pi}{1-y} = \sin(\pi x) + \pi$$
$$y = 1 - \frac{\pi}{\sin(\pi x) + \pi} \quad \text{for } -\infty < x < \infty$$

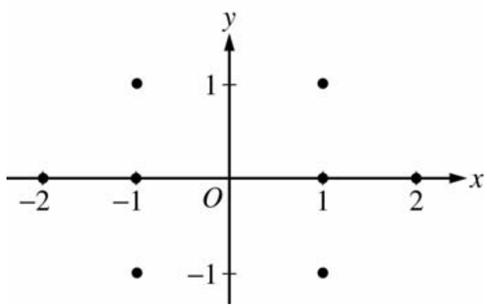
$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration
Note: $0/6$ if no separation of variables

Question 5

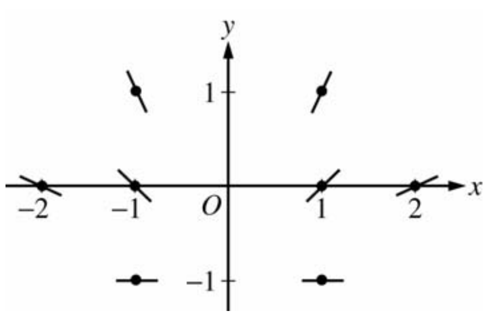
Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1+y}{x}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated. **(Note: Use the axes provided in the pink exam booklet.)**



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

(a)



2 : sign of slope at each point and relative steepness of slope lines in rows and columns

(b) $\dfrac{1}{1+y}\,dy = \dfrac{1}{x}\,dx$

$\ln|1+y| = \ln|x| + K$

$|1+y| = e^{\ln|x| + K}$

$1 + y = C|x|$

$2 = C$

$1 + y = 2|x|$

$y = 2|x| - 1$ and $x < 0$

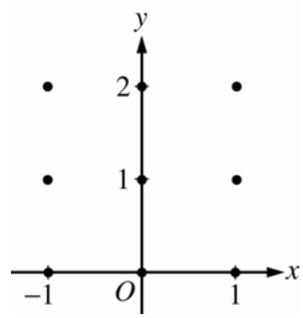
or

$y = -2x - 1$ and $x < 0$

7 :
- 6 :
  - 1 : separates variables
  - 2 : antiderivatives
  - 1 : constant of integration
  - 1 : uses initial condition
  - 1 : solves for y
- Note: max $3/6$ [1-2-0-0-0] if no constant of integration
- Note: $0/6$ if no separation of variables
- 1 : domain

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.

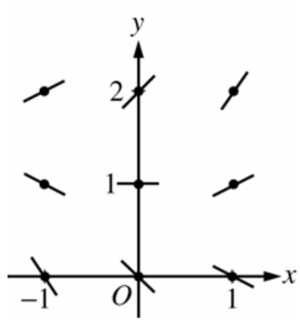


(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy-plane in which all solution curves to the differential equation are concave up.

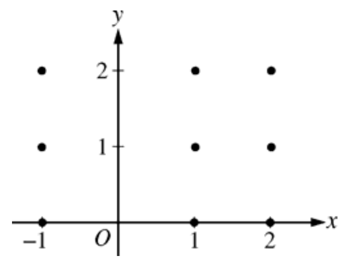
(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

(d) Find the values of the constants m and b, for which $y = mx + b$ is a solution to the differential equation.

(a) 	2 : Sign of slope at each point and relative steepness of slope lines in rows and columns.
(b) $\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2}x + y - \frac{1}{2}$ Solution curves will be concave up on the half-plane above the line $y = -\frac{1}{2}x + \frac{1}{2}$.	3 : { 2 : $\frac{d^2y}{dx^2}$; 1 : description }
(c) $\left.\frac{dy}{dx}\right\|_{(0,1)} = 0 + 1 - 1 = 0$ and $\left.\frac{d^2y}{dx^2}\right\|_{(0,1)} = 0 + 1 - \frac{1}{2} > 0$ Thus, f has a relative minimum at $(0, 1)$.	2 : { 1 : answer; 1 : justification }
(d) Substituting $y = mx + b$ into the differential equation: $m = \frac{1}{2}x + (mx + b) - 1 = \left(m + \frac{1}{2}\right)x + (b - 1)$ Then $0 = m + \frac{1}{2}$ and $m = b - 1$: $m = -\frac{1}{2}$ and $b = \frac{1}{2}$.	2 : { 1 : value for m; 1 : value for b }

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

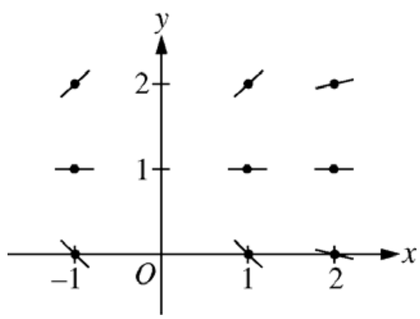


(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)

(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

(c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x\to\infty} f(x)$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

(b) $\frac{1}{y-1}\,dy = \frac{1}{x^2}\,dx$

$\ln|y-1| = -\frac{1}{x} + C$

$|y-1| = e^{-\frac{1}{x}+C}$

$|y-1| = e^{C}e^{-\frac{1}{x}}$

$y - 1 = ke^{-\frac{1}{x}}$, where $k = \pm e^{C}$

$-1 = ke^{-\frac{1}{2}}$

$k = -e^{\frac{1}{2}}$

$f(x) = 1 - e^{\left(\frac{1}{2}-\frac{1}{x}\right)}$, $x > 0$

$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antidifferentiates} \\ 1 : \text{includes constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

(c) $\lim_{x\to\infty} 1 - e^{\left(\frac{1}{2}-\frac{1}{x}\right)} = 1 - \sqrt{e}$

1 : limit

4. Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\dfrac{3x^2+1}{2y}$.
   (a) Find the slope of the graph of f at the point where $x = 1$.
   (b) Write an equation for the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.
   (c) Find $f(x)$ by solving the separable differential equation $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$ with the initial condition $f(1) = 4$.
   (d) Use your solution from part (c) to find $f(1.2)$.

(a) $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$

$\left.\dfrac{dy}{dx}\right|_{\substack{x=1\\y=4}} = \dfrac{3+1}{2\cdot 4} = \dfrac{4}{8} = \dfrac{1}{2}$

1: answer

(b) $y - 4 = \dfrac{1}{2}(x-1)$

$f(1.2) - 4 \approx \dfrac{1}{2}(1.2-1)$

$f(1.2) \approx 0.1 + 4 = 4.1$

2 { 1: equation of tangent line; 1: uses equation to approximate $f(1.2)$ }

(c) $2y\,dy = (3x^2+1)\,dx$

$\int 2y\,dy = \int (3x^2+1)\,dx$

$y^2 = x^3 + x + C$

$4^2 = 1 + 1 + C$

$14 = C$

$y^2 = x^3 + x + 14$

$y = \sqrt{x^3+x+14}$ is branch with point $(1, 4)$

$f(x) = \sqrt{x^3+x+14}$

5 {
- 1: separates variables
- 1: antiderivative of dy term
- 1: antiderivative of dx term
- 1: uses $y = 4$ when $x = 1$ to pick one function out of a family of functions
- 1: solves for y
  0/1 if solving a linear equation in y
  0/1 if no constant of integration
}

Note: max 0/5 if no separation of variables

Note: max 1/5 [1-0-0-0-0] if substitutes value(s) for x, y, or dy/dx before antidifferentiation

(d) $f(1.2) = \sqrt{1.2^3 + 1.2 + 14} \approx 4.114$

1: answer, from student's solution to the given differential equation in (c)

Consider the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$.

(a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \frac{1}{2}$.

(b) Find the domain and range of the function f found in part (a).

(a) $e^{2y}\,dy = 3x^2\,dx$

$$\frac{1}{2}e^{2y} = x^3 + C_1$$

$$e^{2y} = 2x^3 + C$$

$$y = \frac{1}{2}\ln\left(2x^3 + C\right)$$

$$\frac{1}{2} = \frac{1}{2}\ln(0 + C); \quad C = e$$

$$y = \frac{1}{2}\ln(2x^3 + e)$$

6 {
- 1 : separates variables
- 1 : antiderivative of dy term
- 1 : antiderivative of dx term
- 1 : constant of integration
- 1 : uses initial condition $f(0) = \frac{1}{2}$
- 1 : solves for y
- Note: 0/1 if y is not a logarithmic function of x

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

(b) Domain: $2x^3 + e > 0$

$$x^3 > -\frac{1}{2}e$$

$$x > \left(-\frac{1}{2}e\right)^{1/3} = -\left(\frac{1}{2}e\right)^{1/3}$$

Range: $-\infty < y < \infty$

3 {
- 1 : $2x^3 + e > 0$
- 1 : domain
  - Note: 0/1 if 0 is not in the domain
- 1 : range

Note: 0/3 if y is not a logarithmic function of x

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{3-x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. Find the x-coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

(b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

(a) $\frac{dy}{dx} = 0$ when $x = 3$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,-2)} = \left.\frac{-y - y'(3-x)}{y^2}\right|_{(3,-2)} = \frac{1}{2},$$

so f has a local minimum at this point.

or

Because f is continuous for $1 < x < 5$, there is an interval containing $x = 3$ on which $y < 0$. On this interval, $\frac{dy}{dx}$ is negative to the left of $x = 3$ and $\frac{dy}{dx}$ is positive to the right of $x = 3$. Therefore f has a local minimum at $x = 3$.

3 {
- 1 : $x = 3$
- 1 : local minimum
- 1 : justification

(b) $y\,dy = (3 - x)\,dx$

$$\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + C$$

$$8 = 18 - 18 + C\ ;\ C = 8$$

$$y^2 = 6x - x^2 + 16$$

$$y = -\sqrt{6x - x^2 + 16}$$

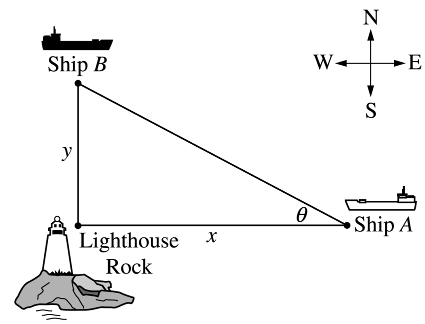
6 {
- 1 : separates variables
- 1 : antiderivative of dy term
- 1 : antiderivative of dx term
- 1 : constant of integration
- 1 : uses initial condition $g(6) = -4$
- 1 : solves for y

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Question 6

Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.



(a) Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km.

(b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

(c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km.

(a) Distance $= \sqrt{3^2 + 4^2} = 5$ km

1 : answer

(b) $r^2 = x^2 + y^2$

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

or explicitly:

$$r = \sqrt{x^2 + y^2}$$

$$\frac{dr}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

At $x = 4$, $y = 3$,

$$\frac{dr}{dt} = \frac{4(-15) + 3(10)}{5} = -6 \text{ km/hr}$$

4 {
- 1 : expression for distance
- 2 : differentiation with respect to t
  - $< -2 >$ chain rule error
- 1 : evaluation

(c) $\tan\theta = \dfrac{y}{x}$

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}$$

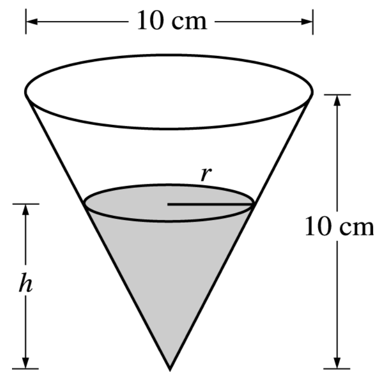
At $x = 4$ and $y = 3$, $\sec\theta = \dfrac{5}{4}$

$$\frac{d\theta}{dt} = \frac{16}{25}\left(\frac{10(4) - (-15)(3)}{16}\right)$$

$$= \frac{85}{25} = \frac{17}{5} \text{ radians/hr}$$

4 {
- 1 : expression for θ in terms of x and y
- 2 : differentiation with respect to t
  - $< -2 >$ chain rule, quotient rule, or transcendental function error
  - note: 0/2 if no trig or inverse trig function
- 1 : evaluation

Question 5



A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

(The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)

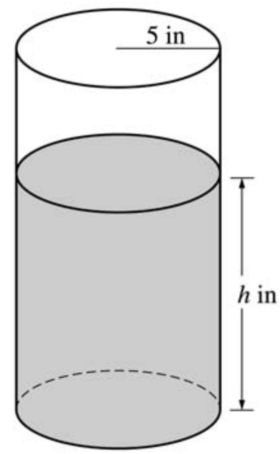
(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

(a) When $h = 5$, $r = \frac{5}{2}$; $V(5) = \frac{1}{3}\pi\left(\frac{5}{2}\right)^2 5 = \frac{125}{12}\pi$ cm^3	1 : V when $h = 5$
(b) $\frac{r}{h} = \frac{5}{10}$, so $r = \frac{1}{2}h$ $V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{1}{12}\pi h^3$; $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$ $\left.\frac{dV}{dt}\right\rvert_{h=5} = \frac{1}{4}\pi(25)\left(-\frac{3}{10}\right) = -\frac{15}{8}\pi$ cm^3/hr OR $\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$; $\frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$ $\left.\frac{dV}{dt}\right\rvert_{h=5,r=\frac{5}{2}} = \frac{1}{3}\pi\left(\left(\frac{25}{4}\right)\left(-\frac{3}{10}\right) + 2\left(\frac{5}{2}\right)5\left(-\frac{3}{20}\right)\right)$ $= -\frac{15}{8}\pi$ cm^3/hr	5 { 1 : $r = \frac{1}{2}h$ in (a) or (b) 1 : { V as a function of one variable in (a) or (b) OR $\frac{dr}{dt}$ 2 : $\frac{dV}{dt}$ < −2 > chain rule or product rule error 1 : evaluation at $h = 5$
(c) $\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt} = -\frac{3}{40}\pi h^2$ $= -\frac{3}{40}\pi(2r)^2 = -\frac{3}{10}\pi r^2 = -\frac{3}{10}\cdot$ area The constant of proportionality is $-\frac{3}{10}$.	2 { 1 : shows $\frac{dV}{dt} = k \cdot$ area 1 : identifies constant of proportionality
units of cm^3 in (a) and cm^3/hr in (b)	**1 : correct units in (a) and (b)**

Question 5



A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Show that $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

(a) $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

3 :
- 1 : $\dfrac{dV}{dt} = -5\pi\sqrt{h}$
- 1 : computes $\dfrac{dV}{dt}$
- 1 : shows result

(b) $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}}\,dh = -\frac{1}{5}\,dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

5 :
- 1 : separates variables
- 1 : antiderivatives
- 1 : constant of integration
- 1 : uses initial condition $h = 17$ when $t = 0$
- 1 : solves for h

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(c) $\left(-\dfrac{1}{10}t + \sqrt{17}\right)^2 = 0$

$$t = 10\sqrt{17}$$

1 : answer

1992 AB6

At time t, $t \geq 0$, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At $t = 0$, the radius of the sphere is 1 and at $t = 15$, the radius is 2. (The volume V of a sphere with a radius r is $V = \frac{4}{3}\pi r^3$.)

(a) Find the radius of the sphere as a function of t.

(b) At what time t will the volume of the sphere be 27 times its volume at $t = 0$?

1992 AB6
Solution

(a) $\dfrac{dV}{dt}=\dfrac{k}{r}$

$\dfrac{dV}{dt}=4\pi r^2\dfrac{dr}{dt}$

$\dfrac{k}{r}=4\pi r^2\dfrac{dr}{dt}$

$k\,dt=4\pi r^3\,dr$

$kt+C=\pi r^4$

At $t=0,\ r=1,$ so $C=\pi$

At $t=15,\ r=2,$ so $15k+\pi=16\pi,\ k=\pi$

$\pi r^4=\pi t+\pi$

$r=\sqrt[4]{t+1}$

(b) At $t=0,\ r=1,$ so $V(0)=\dfrac{4}{3}\pi$

$27V(0)=27\left(\dfrac{4}{3}\pi\right)=36\pi$

$36\pi=\dfrac{4}{3}\pi r^3$

$r=3$

$\sqrt[4]{t+1}=3$

$t=80$

1990 AB4

The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second. (Note: The volume of a sphere with radius r is $\boldsymbol{V} = \frac{4}{3}\pi \boldsymbol{r}^3$.)

(a) At the time when the radius of the sphere is 10 centimeters, what is the rate of increase of its volume?

(b) At the time when the volume of the sphere is 36π cubic centimeters, what is the rate of increase of the area of a cross section through the center of the sphere?

(c) At the time when the volume and the radius of the sphere are increasing at the same numerical rate, what is the radius?

1990 AB4
Solution

(a) $\frac{dV}{dt} = \frac{4}{3} \cdot 3\pi r^2 \frac{dr}{dt}$

Therefore when $r = 10,\ \frac{dr}{dt} = 0.04$

$\frac{dV}{dt} = 4\pi 10^2 \left(0.04\right) = 16\pi \text{ cm}^3\text{/sec}$

(b) $V = 36\pi \Rightarrow 36 = \frac{4}{3} r^3 \Rightarrow r^3 = 27 \Rightarrow r = 3$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

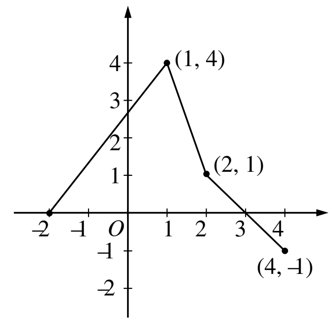
Therefore when $V = 36\pi,\ \frac{dr}{dt} = 0.04$

$\frac{dA}{dt} = 2\pi \cdot 3\left(0.04\right) = \frac{6\pi}{25} = 0.24\pi \text{ cm}^2\text{/sec}$

(c) $\frac{dV}{dt} = \frac{dr}{dt}$

$4\pi r^2 \frac{dr}{dt} = \frac{dr}{dt} \Rightarrow 4\pi r^2 = 1$

Therefore $r^2 = \frac{1}{4\pi} \Rightarrow r = \frac{1}{2\sqrt{\pi}}$ cm



5. The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)\,dt$.
   (a) Compute $g(4)$ and $g(-2)$.
   (b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.
   (c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.
   (d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x–coordinates of points of inflection of the graph of g? Justify your answer.

(a) $g(4) = \int_1^4 f(t)\,dt = \frac{3}{2} + 1 + \frac{1}{2} - \frac{1}{2} = \frac{5}{2}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\frac{1}{2}(12) = -6$

2 { 1: $g(4)$; 1: $g(-2)$ }

(b) $g'(1) = f(1) = 4$

1: answer

(c) g is increasing on $[-2, 3]$ and decreasing on $[3, 4]$.

Therefore, g has absolute minimum at an endpoint of $[-2, 4]$.

Since $g(-2) = -6$ and $g(4) = \frac{5}{2}$,

the absolute minimum value is -6.

3 { 1: interior analysis; 1: endpoint analysis; 1: answer }

(d) One; $x = 1$

On $(-2, 1)$, $g''(x) = f'(x) > 0$

On $(1, 2)$, $g''(x) = f'(x) < 0$

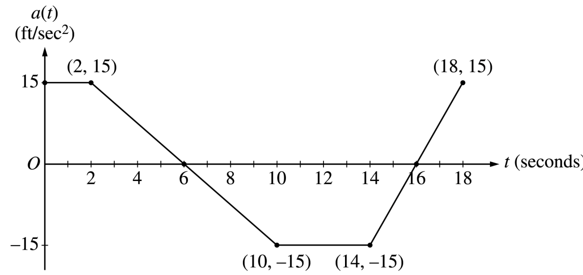
On $(2, 4)$, $g''(x) = f'(x) < 0$

Therefore $(1, g(1))$ is a point of inflection and $(2, g(2))$ is not.

3 { 1: choice of $x = 1$ only; 1: show $(1, g(1))$ is a point of inflection; 1: show $(2, g(2))$ is not a point of inflection }

Question 3

A car is traveling on a straight road with velocity 55 ft/sec at time $t = 0$. For $0 \le t \le 18$ seconds, the car's acceleration $a(t)$, in ft/sec^2, is the piecewise linear function defined by the graph above.



(a) Is the velocity of the car increasing at $t = 2$ seconds? Why or why not?

(b) At what time in the interval $0 \le t \le 18$, other than $t = 0$, is the velocity of the car 55 ft/sec? Why?

(c) On the time interval $0 \le t \le 18$, what is the car's absolute maximum velocity, in ft/sec, and at what time does it occur? Justify your answer.

(d) At what times in the interval $0 \le t \le 18$, if any, is the car's velocity equal to zero? Justify your answer.

(a) Since $v'(2) = a(2)$ and $a(2) = 15 > 0$, the velocity is increasing at $t = 2$.

1 : answer and reason

(b) At time $t = 12$ because

$$v(12) - v(0) = \int_0^{12} a(t)\,dt = 0.$$

$$2 : \begin{cases} 1 : t = 12 \\ 1 : \text{reason} \end{cases}$$

(c) The absolute maximum velocity is 115 ft/sec at $t = 6$.

The absolute maximum must occur at $t = 6$ or at an endpoint.

$$v(6) = 55 + \int_0^6 a(t)\,dt$$
$$= 55 + 2(15) + \frac{1}{2}(4)(15) = 115 > v(0)$$

$$\int_6^{18} a(t)\,dt < 0 \text{ so } v(18) < v(6)$$

4 :
- 1 : $t = 6$
- 1 : absolute maximum velocity
- 1 : identifies $t = 6$ and $t = 18$ as candidates
  or
  indicates that v increases, decreases, then increases
- 1 : eliminates $t = 18$

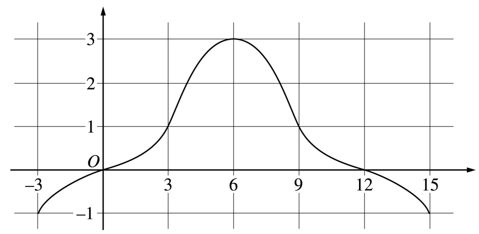
(d) The car's velocity is never equal to 0. The absolute minimum occurs at $t = 16$ where

$$v(16) = 115 + \int_6^{16} a(t)\,dt = 115 - 105 = 10 > 0.$$

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$$

Question 4

The graph of a differentiable function f on the closed interval $[-3, 15]$ is shown in the figure above. The graph of f has a horizontal tangent line at $x = 6$. Let $g(x) = 5 + \int_6^x f(t)\,dt$ for $-3 \le x \le 15$.



Graph of f

(a) Find $g(6)$, $g'(6)$, and $g''(6)$.

(b) On what intervals is g decreasing? Justify your answer.

(c) On what intervals is the graph of g concave down? Justify your answer.

(d) Find a trapezoidal approximation of $\int_{-3}^{15} f(t)\,dt$ using six subintervals of length $\Delta t = 3$.

(a) $g(6) = 5 + \int_6^6 f(t)\,dt = 5$

$g'(6) = f(6) = 3$

$g''(6) = f'(6) = 0$

3 { 1 : $g(6)$; 1 : $g'(6)$; 1 : $g''(6)$ }

(b) g is decreasing on $[-3, 0]$ and $[12, 15]$ since $g'(x) = f(x) < 0$ for $x < 0$ and $x > 12$.

3 { 1 : $[-3, 0]$; 1 : $[12, 15]$; 1 : justification }

(c) The graph of g is concave down on $(6, 15)$ since $g' = f$ is decreasing on this interval.

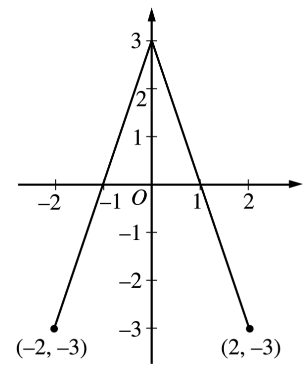
2 { 1 : interval; 1 : justification }

(d) $\frac{3}{2}(-1 + 2(0 + 1 + 3 + 1 + 0) - 1)$
$= 12$

1 : trapezoidal method

Question 4

The graph of the function f shown above consists of two line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.



Graph of f

(a) Find $g(-1)$, $g'(-1)$, and $g''(-1)$.

(b) For what values of x in the open interval $(-2,2)$ is g increasing? Explain your reasoning.

(c) For what values of x in the open interval $(-2,2)$ is the graph of g concave down? Explain your reasoning.

(d) On the axes provided, sketch the graph of g on the closed interval $[-2,2]$.

(a) $g(-1) = \int_0^{-1} f(t)\,dt = -\int_{-1}^{0} f(t)\,dt = -\frac{3}{2}$

$g'(-1) = f(-1) = 0$

$g''(-1) = f'(-1) = 3$

3 { 1 : $g(-1)$; 1 : $g'(-1)$; 1 : $g''(-1)$ }

(b) g is increasing on $-1 < x < 1$ because $g'(x) = f(x) > 0$ on this interval.

2 { 1 : interval; 1 : reason }

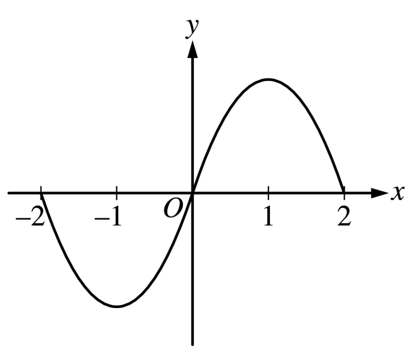
(c) The graph of g is concave down on $0 < x < 2$ because $g''(x) = f'(x) < 0$ on this interval.

or

because $g'(x) = f(x)$ is decreasing on this interval.

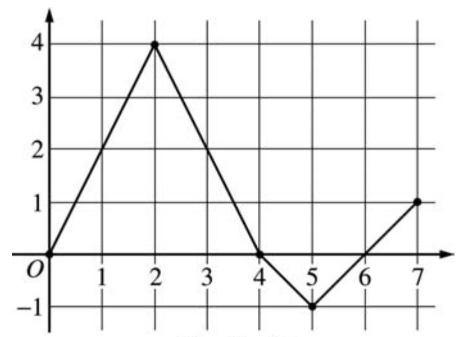
2 { 1 : interval; 1 : reason }

(d)



2 { 1 : $g(-2) = g(0) = g(2) = 0$; 1 : appropriate increasing/decreasing and concavity behavior; $< -1 >$ vertical asymptote }

Question 5



Graph of f

Let f be a function defined on the closed interval $[0,7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.

(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Explain your reasoning.

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.

(a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(4+2) = 3$

$g'(3) = f(3) = 2$

$g''(3) = f'(3) = \frac{0-4}{4-2} = -2$

$3: \begin{cases} 1: g(3) \\ 1: g'(3) \\ 1: g''(3) \end{cases}$

(b) $\frac{g(3)-g(0)}{3} = \frac{1}{3}\int_0^3 f(t)\,dt$

$= \frac{1}{3}\left(\frac{1}{2}(2)(4) + \frac{1}{2}(4+2)\right) = \frac{7}{3}$

$2: \begin{cases} 1: g(3)-g(0) = \int_0^3 f(t)\,dt \\ 1: \text{answer} \end{cases}$

(c) There are two values of c.

We need $\frac{7}{3} = g'(c) = f(c)$

The graph of f intersects the line $y = \frac{7}{3}$ at two places between 0 and 3.

$2: \begin{cases} 1: \text{answer of } 2 \\ 1: \text{reason} \end{cases}$

Note: 1/2 if answer is 1 by MVT

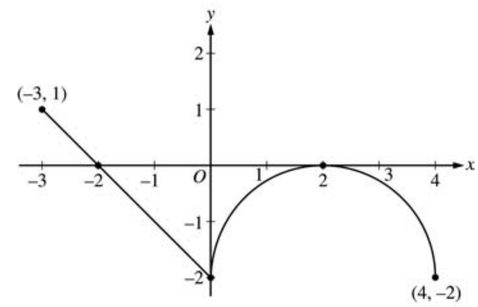
(d) $x = 2$ and $x = 5$

because $g' = f$ changes from increasing to decreasing at $x = 2$, and from decreasing to increasing at $x = 5$.

$2: \begin{cases} 1: x = 2 \text{ and } x = 5 \text{ only} \\ 1: \text{justification} \\ \quad (\text{ignore discussion at } x = 4) \end{cases}$

Question 4

Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown above.



Graph of f'

(a) On what intervals, if any, is f increasing? Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

(c) Find an equation for the line tangent to the graph of f at the point $(0, 3)$.

(d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.

(a) The function f is increasing on $[-3, -2]$ since $f' > 0$ for $-3 \le x < -2$.

$2: \begin{cases} 1: \text{interval} \\ 1: \text{reason} \end{cases}$

(b) $x = 0$ and $x = 2$

f' changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 2$

$2: \begin{cases} 1: x = 0 \text{ and } x = 2 \text{ only} \\ 1: \text{justification} \end{cases}$

(c) $f'(0) = -2$

Tangent line is $y = -2x + 3$.

$1:$ equation

(d) $f(0) - f(-3) = \int_{-3}^{0} f'(t)\,dt$

$= \frac{1}{2}(1)(1) - \frac{1}{2}(2)(2) = -\frac{3}{2}$

$f(-3) = f(0) + \frac{3}{2} = \frac{9}{2}$

$f(4) - f(0) = \int_{0}^{4} f'(t)\,dt$

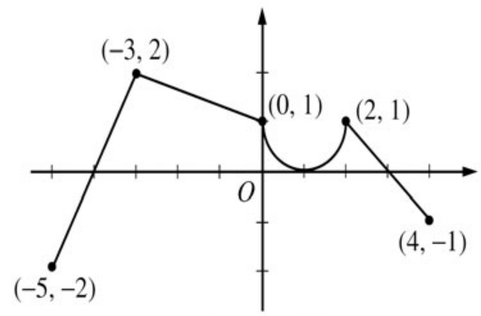
$= -\left(8 - \frac{1}{2}(2)^2\pi\right) = -8 + 2\pi$

$f(4) = f(0) - 8 + 2\pi = -5 + 2\pi$

$4: \begin{cases} 1: \pm\left(\frac{1}{2} - 2\right) \text{ (difference of areas of triangles)} \\ 1: \text{answer for } f(-3) \text{ using FTC} \\ 1: \pm\left(8 - \frac{1}{2}(2)^2\pi\right) \text{ (area of rectangle} - \text{area of semicircle)} \\ 1: \text{answer for } f(4) \text{ using FTC} \end{cases}$

Question 5

The graph of the function f shown above consists of a semicircle and three line segments. Let g be the function given by $g(x) = \int_{-3}^{x} f(t)\,dt.$



Graph of f

(a) Find $g(0)$ and $g'(0)$.

(b) Find all values of x in the open interval $(-5, 4)$ at which g attains a relative maximum. Justify your answer.

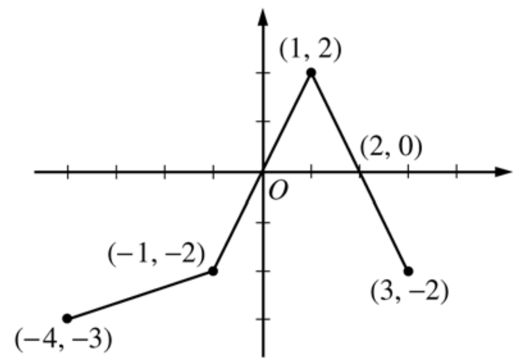
(c) Find the absolute minimum value of g on the closed interval $[-5, 4]$. Justify your answer.

(d) Find all values of x in the open interval $(-5, 4)$ at which the graph of g has a point of inflection.

(a) $g(0) = \int_{-3}^{0} f(t)\,dt = \frac{1}{2}(3)(2+1) = \frac{9}{2}$ $g'(0) = f(0) = 1$	$2 : \begin{cases} 1 : g(0) \\ 1 : g'(0) \end{cases}$
(b) g has a relative maximum at $x = 3$. This is the only x-value where $g' = f$ changes from positive to negative.	$2 : \begin{cases} 1 : x = 3 \\ 1 : \text{justification} \end{cases}$
(c) The only x-value where f changes from negative to positive is $x = -4$. The other candidates for the location of the absolute minimum value are the endpoints. $g(-5) = 0$ $g(-4) = \int_{-3}^{-4} f(t)\,dt = -1$ $g(4) = \frac{9}{2} + \left(2 - \frac{\pi}{2}\right) = \frac{13 - \pi}{2}$ So the absolute minimum value of g is -1.	$3 : \begin{cases} 1 : \text{identifies } x = -4 \text{ as a candidate} \\ 1 : g(-4) = -1 \\ 1 : \text{justification and answer} \end{cases}$
(d) $x = -3, 1, 2$	2 : correct values $\langle -1 \rangle$ each missing or extra value

Question 4

The graph of the function f above consists of three line segments.



Graph of f

(a) Let g be the function given by $g(x) = \int_{-4}^{x} f(t)\,dt$.

For each of $g(-1)$, $g'(-1)$, and $g''(-1)$, find the value or state that it does not exist.

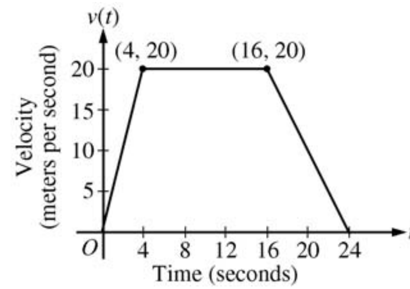
(b) For the function g defined in part (a), find the x-coordinate of each point of inflection of the graph of g on the open interval $-4 < x < 3$. Explain your reasoning.

(c) Let h be the function given by $h(x) = \int_{x}^{3} f(t)\,dt$. Find all values of x in the closed interval $-4 \le x \le 3$ for which $h(x) = 0$.

(d) For the function h defined in part (c), find all intervals on which h is decreasing. Explain your reasoning.

(a) $g(-1) = \int_{-4}^{-1} f(t)\,dt = -\frac{1}{2}(3)(5) = -\frac{15}{2}$ $g'(-1) = f(-1) = -2$ $g''(-1)$ does not exist because f is not differentiable at $x = -1$.	$3 : \begin{cases} 1 : g(-1) \\ 1 : g'(-1) \\ 1 : g''(-1) \end{cases}$
(b) $x = 1$ $g' = f$ changes from increasing to decreasing at $x = 1$.	$2 : \begin{cases} 1 : x = 1 \text{ (only)} \\ 1 : \text{reason} \end{cases}$
(c) $x = -1,\ 1,\ 3$	2 : correct values $\langle -1 \rangle$ each missing or extra value
(d) h is decreasing on $[0, 2]$ $h' = -f < 0$ when $f > 0$	$2 : \begin{cases} 1 : \text{interval} \\ 1 : \text{reason} \end{cases}$

Question 5



A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.

(a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

(b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

(c) Let $a(t)$ be the car's acceleration at time t, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

(d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

(a) $$\int_0^{24} v(t)\,dt = \frac{1}{2}(4)(20) + (12)(20) + \frac{1}{2}(8)(20) = 360$$

The car travels 360 meters in these 24 seconds.

$2: \begin{cases} 1: \text{value} \\ 1: \text{meaning with units} \end{cases}$

(b) $v'(4)$ does not exist because

$$\lim_{t\to 4^-}\left(\frac{v(t)-v(4)}{t-4}\right) = 5 \ne 0 = \lim_{t\to 4^+}\left(\frac{v(t)-v(4)}{t-4}\right).$$

$$v'(20) = \frac{20-0}{16-24} = -\frac{5}{2}\ \text{m/sec}^2$$

$3: \begin{cases} 1: v'(4) \text{ does not exist, with explanation} \\ 1: v'(20) \\ 1: \text{units} \end{cases}$

(c) $$a(t) = \begin{cases} 5 & \text{if } 0 < t < 4 \\ 0 & \text{if } 4 < t < 16 \\ -\frac{5}{2} & \text{if } 16 < t < 24 \end{cases}$$

$a(t)$ does not exist at $t = 4$ and $t = 16$.

$2: \begin{cases} 1: \text{finds the values } 5, 0, -\frac{5}{2} \\ 1: \text{identifies constants with correct intervals} \end{cases}$

(d) The average rate of change of v on $[8, 20]$ is

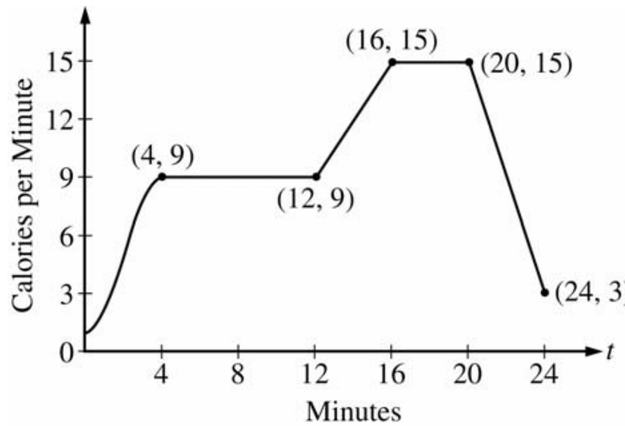
$$\frac{v(20)-v(8)}{20-8} = -\frac{5}{6}\ \text{m/sec}^2.$$

No, the Mean Value Theorem does not apply to v on $[8, 20]$ because v is not differentiable at $t = 16$.

$2: \begin{cases} 1: \text{average rate of change of } v \text{ on } [8, 20] \\ 1: \text{answer with explanation} \end{cases}$

Question 4

The rate, in calories per minute, at which a person using an exercise machine burns calories is modeled by the function f. In the figure above, $f(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$ for $0 \le t \le 4$ and f is piecewise linear for $4 \le t \le 24$.



(a) Find $f'(22)$. Indicate units of measure.

(b) For the time interval $0 \le t \le 24$, at what time t is f increasing at its greatest rate? Show the reasoning that supports your answer.

(c) Find the total number of calories burned over the time interval $6 \le t \le 18$ minutes.

(d) The setting on the machine is now changed so that the person burns $f(t) + c$ calories per minute. For this setting, find c so that an average of 15 calories per minute is burned during the time interval $6 \le t \le 18$.

(a) $f'(22) = \dfrac{15-3}{20-24} = -3$ calories/min/min

1 : $f'(22)$ and units

(b) f is increasing on $[0, 4]$ and on $[12, 16]$.

On $(12, 16)$, $f'(t) = \dfrac{15-9}{16-12} = \dfrac{3}{2}$ since f has constant slope on this interval.

On $(0, 4)$, $f'(t) = -\frac{3}{4}t^2 + 3t$ and

$f''(t) = -\frac{3}{2}t + 3 = 0$ when $t = 2$. This is where f' has a maximum on $[0, 4]$ since $f'' > 0$ on $(0, 2)$ and $f'' < 0$ on $(2, 4)$.

On $[0, 24]$, f is increasing at its greatest rate when $t = 2$ because $f'(2) = 3 > \frac{3}{2}$.

4 :
- 1 : f' on $(0, 4)$
- 1 : shows f' has a max at $t = 2$ on $(0, 4)$
- 1 : shows for $12 < t < 16$, $f'(t) < f'(2)$
- 1 : answer

(c) $\displaystyle\int_6^{18} f(t)\,dt = 6(9) + \frac{1}{2}(4)(9+15) + 2(15)$
$= 132$ calories

2 :
- 1 : method
- 1 : answer

(d) We want $\displaystyle\frac{1}{12}\int_6^{18} (f(t)+c)\,dt = 15$.

This means $132 + 12c = 15(12)$. So, $c = 4$.

OR

Currently, the average is $\dfrac{132}{12} = 11$ calories/min.

Adding c to $f(t)$ will shift the average by c.
So $c = 4$ to get an average of 15 calories/min.

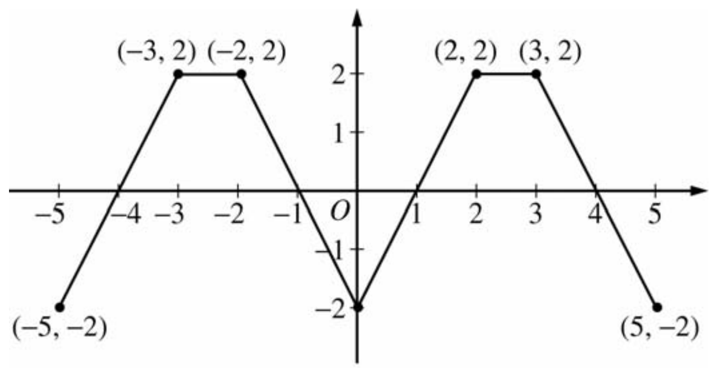
2 :
- 1 : setup
- 1 : value of c

Question 3

The graph of the function f shown above consists of six line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt.$

(a) Find $g(4)$, $g'(4)$, and $g''(4)$.

(b) Does g have a relative minimum, a relative maximum, or neither at $x = 1$? Justify your answer.



Graph of f

(c) Suppose that f is defined for all real numbers x and is periodic with a period of length 5. The graph above shows two periods of f. Given that $g(5) = 2$, find $g(10)$ and write an equation for the line tangent to the graph of g at $x = 108$.

(a) $g(4) = \int_0^4 f(t)\,dt = 3$

$g'(4) = f(4) = 0$

$g''(4) = f'(4) = -2$

$3 : \begin{cases} 1 : g(4) \\ 1 : g'(4) \\ 1 : g''(4) \end{cases}$

(b) g has a relative minimum at $x = 1$ because $g' = f$ changes from negative to positive at $x = 1$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $g(0) = 0$ and the function values of g increase by 2 for every increase of 5 in x.

$g(10) = 2g(5) = 4$

$$g(108) = \int_0^{105} f(t)\,dt + \int_{105}^{108} f(t)\,dt$$
$$= 21g(5) + g(3) = 44$$

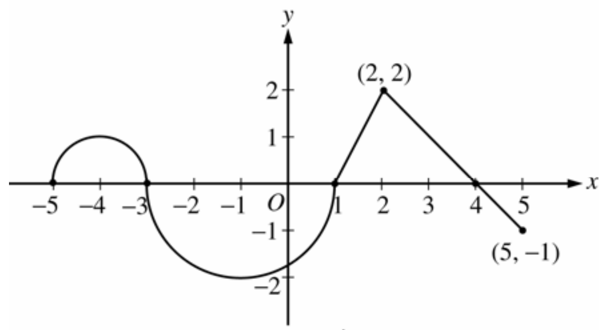
$g'(108) = f(108) = f(3) = 2$

An equation for the line tangent to the graph of g at $x = 108$ is $y - 44 = 2(x - 108)$.

$4 : \begin{cases} 1 : g(10) \\ 3 : \begin{cases} 1 : g(108) \\ 1 : g'(108) \\ 1 : \text{equation of tangent line} \end{cases} \end{cases}$

Question 4

Let f be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of f', the derivative of f, consists of two semicircles and two line segments, as shown above.



Graph of f'

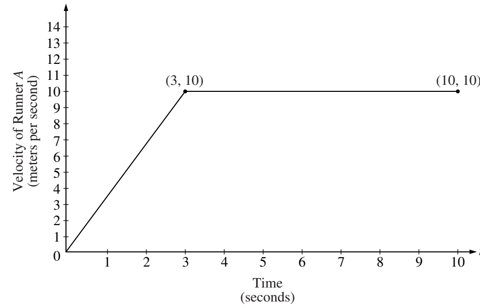
(a) For $-5 < x < 5$, find all values x at which f has a relative maximum. Justify your answer.

(b) For $-5 < x < 5$, find all values x at which the graph of f has a point of inflection. Justify your answer.

(c) Find all intervals on which the graph of f is concave up and also has positive slope. Explain your reasoning.

(d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

(a) $f'(x) = 0$ at $x = -3,\ 1,\ 4$ f' changes from positive to negative at -3 and 4. Thus, f has a relative maximum at $x = -3$ and at $x = 4$.	$2 : \begin{cases} 1 : x\text{-values} \\ 1 : \text{justification} \end{cases}$
(b) f' changes from increasing to decreasing, or vice versa, at $x = -4,\ -1,$ and 2. Thus, the graph of f has points of inflection when $x = -4,\ -1,$ and 2.	$2 : \begin{cases} 1 : x\text{-values} \\ 1 : \text{justification} \end{cases}$
(c) The graph of f is concave up with positive slope where f' is increasing and positive: $-5 < x < -4$ and $1 < x < 2$.	$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{explanation} \end{cases}$
(d) Candidates for the absolute minimum are where f' changes from negative to positive (at $x = 1$) and at the endpoints ($x = -5, 5$). $f(-5) = 3 + \int_1^{-5} f'(x)\,dx = 3 - \frac{\pi}{2} + 2\pi > 3$ $f(1) = 3$ $f(5) = 3 + \int_1^5 f'(x)\,dx = 3 + \frac{3 \cdot 2}{2} - \frac{1}{2} > 3$ The absolute minimum value of f on $[-5, 5]$ is $f(1) = 3$.	$3 : \begin{cases} 1 : \text{identifies } x = 1 \text{ as a candidate} \\ 1 : \text{considers endpoints} \\ 1 : \text{value and explanation} \end{cases}$



Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \dfrac{24t}{2t+3}$.

(a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.

(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

(c) Find the total distance run by Runner A and the total distance run by Runner B over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

(a) Runner A: velocity $= \dfrac{10}{3} \cdot 2 = \dfrac{20}{3}$

$= 6.666$ or 6.667 meters/sec

Runner B: $v(2) = \dfrac{48}{7} = 6.857$ meters/sec

2 { 1 : velocity for Runner A; 1 : velocity for Runner B }

(b) Runner A: acceleration $= \dfrac{10}{3} = 3.333$ meters/sec^2

Runner B: $a(2) = v'(2) = \left.\dfrac{72}{(2t+3)^2}\right|_{t=2}$

$= \dfrac{72}{49} = 1.469$ meters/sec^2

2 { 1 : acceleration for Runner A; 1 : acceleration for Runner B }

(c) Runner A: distance $= \dfrac{1}{2}(3)(10) + 7(10) = 85$ meters

Runner B: distance $= \displaystyle\int_0^{10} \frac{24t}{2t+3}\,dt = 83.336$ meters

4 {
- 2 : distance for Runner A
  - 1 : method
  - 1 : answer
- 2 : distance for Runner B
  - 1 : integral
  - 1 : answer

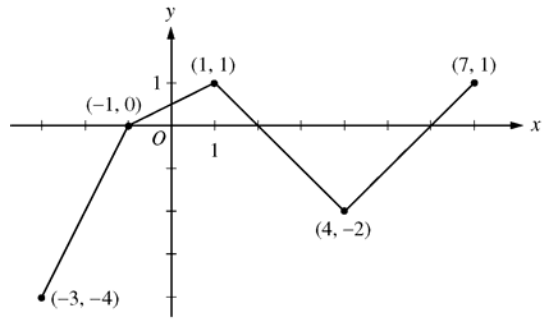
}

(units) meters/sec in part (a), meters/sec^2 in part (b), and meters in part (c), or equivalent.

1: units

Question 5

Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.



Graph of g'

(a) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

(b) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

(c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

(a) g' changes from increasing to decreasing at $x = 1$;
g' changes from decreasing to increasing at $x = 4$.

Points of inflection for the graph of $y = g(x)$ occur at $x = 1$ and $x = 4$.

$2 : \begin{cases} 1 : x\text{-values} \\ 1 : \text{justification} \end{cases}$

(b) The only sign change of g' from positive to negative in the interval is at $x = 2$.

$$g(-3) = 5 + \int_2^{-3} g'(x)\,dx = 5 + \left(-\frac{3}{2}\right) + 4 = \frac{15}{2}$$
$$g(2) = 5$$
$$g(7) = 5 + \int_2^{7} g'(x)\,dx = 5 + (-4) + \frac{1}{2} = \frac{3}{2}$$

The maximum value of g for $-3 \le x \le 7$ is $\frac{15}{2}$.

$3 : \begin{cases} 1 : \text{identifies } x = 2 \text{ as a candidate} \\ 1 : \text{considers endpoints} \\ 1 : \text{maximum value and justification} \end{cases}$

(c) $$\frac{g(7) - g(-3)}{7 - (-3)} = \frac{\frac{3}{2} - \frac{15}{2}}{10} = -\frac{3}{5}$$

$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer} \end{cases}$

(d) $$\frac{g'(7) - g'(-3)}{7 - (-3)} = \frac{1 - (-4)}{10} = \frac{1}{2}$$

No, the MVT does *not* guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$.

$2 : \begin{cases} 1 : \text{average value of } g'(x) \\ 1 : \text{answer “No” with reason} \end{cases}$

1998 AP Calculus AB Scoring Guidelines

6. Consider the curve defined by $2y^3 + 6x^2y - 12x^2 + 6y = 1$.

   (a) Show that $\dfrac{dy}{dx} = \dfrac{4x - 2xy}{x^2 + y^2 + 1}$.

   (b) Write an equation of each horizontal tangent line to the curve.

   (c) The line through the origin with slope -1 is tangent to the curve at point P. Find the x– and y–coordinates of point P.

(a) $6y^2\dfrac{dy}{dx} + 6x^2\dfrac{dy}{dx} + 12xy - 24x + 6\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}(6y^2 + 6x^2 + 6) = 24x - 12xy$

$\dfrac{dy}{dx} = \dfrac{24x - 12xy}{6x^2 + 6y^2 + 6} = \dfrac{4x - 2xy}{x^2 + y^2 + 1}$

2 {
- 1: implicit differentiation
- 1: verifies expression for $\dfrac{dy}{dx}$

(b) $\dfrac{dy}{dx} = 0$

$4x - 2xy = 2x(2 - y) = 0$

$x = 0$ or $y = 2$

When $x = 0$, $2y^3 + 6y = 1$; $y = 0.165$

There is no point on the curve with y coordinate of 2.

$y = 0.165$ is the equation of the only horizontal tangent line.

4 {
- 1: sets $\dfrac{dy}{dx} = 0$
- 1: solves $\dfrac{dy}{dx} = 0$
- 1: uses solutions for x to find equations of horizontal tangent lines
- 1: verifies which solutions for y yield equations of horizontal tangent lines

Note: max 1/4 [1-0-0-0] if $dy/dx = 0$ is not of the form $g(x, y)/h(x, y) = 0$ with solutions for both x and y

(c) $y = -x$ is equation of the line.

$2(-x)^3 + 6x^2(-x) - 12x^2 + 6(-x) = 1$

$-8x^3 - 12x^2 - 6x - 1 = 0$

$x = -1/2, \quad y = 1/2$

–or–

$\dfrac{dy}{dx} = -1$

$4x - 2xy = -x^2 - y^2 - 1$

$4x + 2x^2 = -x^2 - x^2 - 1$

$4x^2 + 4x + 1 = 0$

$x = -1/2, \quad y = 1/2$

3 {
- 1: $y = -x$
- 1: substitutes $y = -x$ into equation of curve
- 1: solves for x and y

–or–

3 {
- 1: sets $\dfrac{dy}{dx} = -1$
- 1: substitutes $y = -x$ into $\dfrac{dy}{dx}$
- 1: solves for x and y

Note: max 2/3 [1-1-0] if importing incorrect derivative from part (a)

Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

(a) $y^2 + 2xy\dfrac{dy}{dx} - 3x^2y - x^3\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}\left(2xy - x^3\right) = 3x^2y - y^2$

$\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$

$2\begin{cases} 1: \text{implicit differentiation} \\ 1: \text{verifies expression for } \dfrac{dy}{dx} \end{cases}$

(b) When $x = 1$, $y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3,\ y = -2$

At $(1,3)$, $\dfrac{dy}{dx} = \dfrac{9-9}{6-1} = 0$

Tangent line equation is $y = 3$

At $(1,-2)$, $\dfrac{dy}{dx} = \dfrac{-6-4}{-4-1} = \dfrac{-10}{-5} = 2$

Tangent line equation is $y + 2 = 2(x - 1)$

$4\begin{cases} 1: y^2 - y = 6 \\ 1: \text{solves for } y \\ 2: \text{tangent lines} \end{cases}$

Note: 0/4 if not solving an equation of the form $y^2 - y = k$

(c) Tangent line is vertical when $2xy - x^3 = 0$

$x\left(2y - x^2\right) = 0$ gives $x = 0$ or $y = \dfrac{1}{2}x^2$

There is no point on the curve with x-coordinate 0.

When $y = \dfrac{1}{2}x^2$, $\dfrac{1}{4}x^5 - \dfrac{1}{2}x^5 = 6$

$-\dfrac{1}{4}x^5 = 6$

$x = \sqrt[5]{-24}$

$3\begin{cases} 1: \text{sets denominator of } \dfrac{dy}{dx} \text{ equal to 0} \\ 1: \text{substitutes } y = \dfrac{1}{2}x^2 \text{ or } x = \pm\sqrt{2y} \\ \quad \text{into the equation for the curve} \\ 1: \text{solves for } x\text{-coordinate} \end{cases}$

Question 6

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

(a) $$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}(6 - 2x) - 2y^2 = 2y^3(6 - 2x)^2 - 2y^2$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(3, \frac{1}{4}\right)} = 0 - 2\left(\frac{1}{4}\right)^2 = -\frac{1}{8}$$

3 : $\begin{cases} 2 : \frac{d^2y}{dx^2} \\ \quad < -2 > \text{ product rule or chain rule error} \\ 1 : \text{value at } \left(3, \frac{1}{4}\right) \end{cases}$

(b) $$\frac{1}{y^2}\,dy = (6 - 2x)\,dx$$

$$-\frac{1}{y} = 6x - x^2 + C$$

$$-4 = 18 - 9 + C = 9 + C$$

$$C = -13$$

$$y = \frac{1}{x^2 - 6x + 13}$$

6 : $\begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivative of } dy \text{ term} \\ 1 : \text{antiderivative of } dx \text{ term} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } f(3) = \frac{1}{4} \\ 1 : \text{solves for } y \end{cases}$

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Question 4

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\dfrac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P ? Justify your answer.

(a)
$$2x + 8yy' = 3y + 3xy'$$
$$(8y - 3x)y' = 3y - 2x$$
$$y' = \frac{3y - 2x}{8y - 3x}$$

$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$

(b) $\dfrac{3y - 2x}{8y - 3x} = 0;\ 3y - 2x = 0$

When $x = 3,\ 3y = 6$
$y = 2$

$3^2 + 4 \cdot 2^2 = 25$ and $7 + 3 \cdot 3 \cdot 2 = 25$

Therefore, $P = (3, 2)$ is on the curve and the slope is 0 at this point.

$3 : \begin{cases} 1 : \dfrac{dy}{dx} = 0 \\ 1 : \text{shows slope is 0 at } (3, 2) \\ 1 : \text{shows } (3, 2) \text{ lies on curve} \end{cases}$

(c) $\dfrac{d^2y}{dx^2} = \dfrac{(8y - 3x)(3y' - 2) - (3y - 2x)(8y' - 3)}{(8y - 3x)^2}$

At $P = (3, 2)$, $\dfrac{d^2y}{dx^2} = \dfrac{(16 - 9)(-2)}{(16 - 9)^2} = -\dfrac{2}{7}$.

Since $y' = 0$ and $y'' < 0$ at P, the curve has a local maximum at P.

$4 : \begin{cases} 2 : \dfrac{d^2y}{dx^2} \\ 1 : \text{value of } \dfrac{d^2y}{dx^2} \text{ at } (3, 2) \\ 1 : \text{conclusion with justification} \end{cases}$

Question 5

Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$.

(b) Find all points (x, y) on the curve where the line tangent to the curve has slope $\frac{1}{2}$.

(c) Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.

(d) Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $\dfrac{dy}{dt} = 6$. Find the value of $\dfrac{dx}{dt}$ at time $t = 5$.

(a) $2yy' = y + xy'$
$(2y - x)y' = y$
$y' = \dfrac{y}{2y - x}$

$2: \begin{cases} 1: \text{implicit differentiation} \\ 1: \text{solves for } y' \end{cases}$

(b) $\dfrac{y}{2y - x} = \dfrac{1}{2}$
$2y = 2y - x$
$x = 0$
$y = \pm\sqrt{2}$
$(0, \sqrt{2}), (0, -\sqrt{2})$

$2: \begin{cases} 1: \dfrac{y}{2y - x} = \dfrac{1}{2} \\ 1: \text{answer} \end{cases}$

(c) $\dfrac{y}{2y - x} = 0$
$y = 0$
The curve has no horizontal tangent since $0^2 \neq 2 + x \cdot 0$ for any x.

$2: \begin{cases} 1: y = 0 \\ 1: \text{explanation} \end{cases}$

(d) When $y = 3$, $3^2 = 2 + 3x$ so $x = \dfrac{7}{3}$.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{y}{2y - x} \cdot \frac{dx}{dt}$$

At $t = 5$, $6 = \dfrac{3}{6 - \frac{7}{3}} \cdot \dfrac{dx}{dt} = \dfrac{9}{11} \cdot \dfrac{dx}{dt}$

$$\left.\frac{dx}{dt}\right|_{t=5} = \frac{22}{3}$$

$3: \begin{cases} 1: \text{solves for } x \\ 1: \text{chain rule} \\ 1: \text{answer} \end{cases}$

1. A particle moves along the y–axis with velocity given by $v(t) = t\sin(t^2)$ for $t \geq 0$.
   (a) In which direction (up or down) is the particle moving at time $t = 1.5$? Why?
   (b) Find the acceleration of the particle at time $t = 1.5$. Is the velocity of the particle increasing at $t = 1.5$? Why or why not?
   (c) Given that $y(t)$ is the position of the particle at time t and that $y(0) = 3$, find $y(2)$.
   (d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

(a) $v(1.5) = 1.5\sin(1.5^2) = 1.167$

Up, because $v(1.5) > 0$

1: answer and reason

(b) $a(t) = v'(t) = \sin t^2 + 2t^2 \cos t^2$

$a(1.5) = v'(1.5) = -2.048$ or -2.049

No; v is decreasing at 1.5 because $v'(1.5) < 0$

2 $\begin{cases} \text{1: } a(1.5) \\ \text{1: conclusion and reason} \end{cases}$

(c) $y(t) = \int v(t)\,dt$

$= \int t\sin t^2\,dt = -\dfrac{\cos t^2}{2} + C$

$y(0) = 3 = -\dfrac{1}{2} + C \implies C = \dfrac{7}{2}$

$y(t) = -\dfrac{1}{2}\cos t^2 + \dfrac{7}{2}$

$y(2) = -\dfrac{1}{2}\cos 4 + \dfrac{7}{2} = 3.826$ or 3.827

3 $\begin{cases} \text{1: } y(t) = \int v(t)\,dt \\ \text{1: } y(t) = -\frac{1}{2}\cos t^2 + C \\ \text{1: } y(2) \end{cases}$

(d) distance $= \int_0^2 |v(t)|\,dt = 1.173$

or

$v(t) = t\sin t^2 = 0$

$t = 0$ or $t = \sqrt{\pi} \approx 1.772$

$y(0) = 3\,;\ \ y(\sqrt{\pi}) = 4\,;\ \ y(2) = 3.826$ or 3.827

$[y(\sqrt{\pi}) - y(0)] + [y(\sqrt{\pi}) - y(2)]$

$= 1.173$ or 1.174

3
- 1: limits of 0 and 2 on an integral of $v(t)$ or $|v(t)|$
  or
  uses $y(0)$ and $y(2)$ to compute distance
- 1: handles change of direction at student's turning point
- 1: answer
  0/1 if incorrect turning point

Question 3

A particle moves along the x-axis so that its velocity v at any time t, for $0 \le t \le 16$, is given by $v(t) = e^{2\sin t} - 1$. At time $t = 0$, the particle is at the origin.

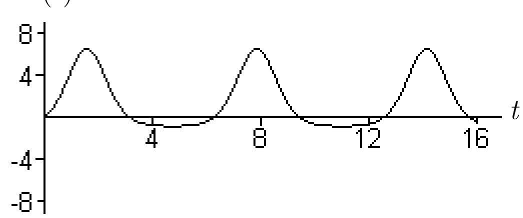
(a) On the axes provided, sketch the graph of $v(t)$ for $0 \le t \le 16$.

(b) During what intervals of time is the particle moving to the left? Give a reason for your answer.

(c) Find the total distance traveled by the particle from $t = 0$ to $t = 4$.

(d) Is there any time t, $0 < t \le 16$, at which the particle returns to the origin? Justify your answer.

(a)



1 : graph
- three "humps"
- periodic behavior
- starts at origin
- reasonable relative max and min values

(b) Particle is moving to the left when

$v(t) < 0$, i.e. $e^{2\sin t} < 1$.

$(\pi, 2\pi)$, $(3\pi, 4\pi)$ and $(5\pi, 16]$

3 { 2 : intervals
  $< -1 >$ each missing or incorrect interval
1 : reason

(c) $\int_0^4 |v(t)|\,dt = 10.542$

or

$v(t) = e^{2\sin t} - 1 = 0$

$t = 0$ or $t = \pi$

$x(\pi) = \int_0^{\pi} v(t)\,dt = 10.10656$

$x(4) = \int_0^{4} v(t)\,dt = 9.67066$

$|x(\pi) - x(0)| + |x(4) - x(\pi)|$

$= 10.542$

3 { 1 : limits of 0 and 4 on an integral of
  $v(t)$ or $|v(t)|$
  or
  uses $x(0)$ and $x(4)$ to compute distance
1 : handles change of direction at student's turning point
1 : answer
  note: 0/1 if incorrect turning point

(d) There is no such time because $\int_0^T v(t)\,dt > 0$ for all $T > 0$.

2 { 1 : no such time
1 : reason

Question 3

An object moves along the x-axis with initial position $x(0) = 2$. The velocity of the object at time $t \geq 0$ is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$.

(a) What is the acceleration of the object at time $t = 4$?

(b) Consider the following two statements.

Statement I: For $3 < t < 4.5$, the velocity of the object is decreasing.

Statement II: For $3 < t < 4.5$, the speed of the object is increasing.

Are either or both of these statements correct? For each statement provide a reason why it is correct or not correct.

(c) What is the total distance traveled by the object over the time interval $0 \leq t \leq 4$?

(d) What is the position of the object at time $t = 4$?

(a) $a(4) = v'(4) = \frac{\pi}{3}\cos\left(\frac{4\pi}{3}\right)$ $= -\frac{\pi}{6}$ or -0.523 or -0.524	1 : answer
(b) On $3 < t < 4.5$: $a(t) = v'(t) = \frac{\pi}{3}\cos\left(\frac{\pi}{3}t\right) < 0$ Statement I is correct since $a(t) < 0$. Statement II is correct since $v(t) < 0$ and $a(t) < 0$.	3 { 1 : I correct, with reason 1 : II correct 1 : reason for II
(c) Distance $= \int_0^4 \lvert v(t)\rvert\, dt = 2.387$ OR $x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + \frac{3}{\pi} + 2$ $x(0) = 2$ $x(4) = 2 + \frac{9}{2\pi} = 3.43239$ $v(t) = 0$ when $t = 3$ $x(3) = \frac{6}{\pi} + 2 = 3.90986$ $\lvert x(3) - x(0)\rvert + \lvert x(4) - x(3)\rvert = \frac{15}{2\pi} = 2.387$	3 { 1 : { limits of 0 and 4 on an integral of $v(t)$ or $\lvert v(t)\rvert$ or uses $x(0)$ and $x(4)$ to compute distance 1 : handles change of direction at student's turning point 1 : answer 0/1 if incorrect turning point or no turning point
(d) $x(4) = x(0) + \int_0^4 v(t)\, dt = 3.432$ OR $x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + \frac{3}{\pi} + 2$ $x(4) = 2 + \frac{9}{2\pi} = 3.432$	2 { 1 : integral 1 : answer OR 2 { 1 : $x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + C$ 1 : answer 0/1 if no constant of integration

Question 4

A particle moves along the x-axis with velocity at time $t \geq 0$ given by $v(t) = -1 + e^{1-t}$.

(a) Find the acceleration of the particle at time $t = 3$.

(b) Is the speed of the particle increasing at time $t = 3$? Give a reason for your answer.

(c) Find all values of t at which the particle changes direction. Justify your answer.

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

(a) $a(t) = v'(t) = -e^{1-t}$

$a(3) = -e^{-2}$

$2 : \begin{cases} 1 : v'(t) \\ 1 : a(3) \end{cases}$

(b) $a(3) < 0$

$v(3) = -1 + e^{-2} < 0$

Speed is increasing since $v(3) < 0$ and $a(3) < 0$.

1 : answer with reason

(c) $v(t) = 0$ when $1 = e^{1-t}$, so $t = 1$.

$v(t) > 0$ for $t < 1$ and $v(t) < 0$ for $t > 1$.

Therefore, the particle changes direction at $t = 1$.

$2 : \begin{cases} 1 : \text{solves } v(t) = 0 \text{ to get } t = 1 \\ 1 : \text{justifies change in direction at } t = 1 \end{cases}$

(d)
$$\begin{aligned} \text{Distance} &= \int_0^3 |v(t)|\,dt \\ &= \int_0^1 \left(-1 + e^{1-t}\right) dt + \int_1^3 \left(1 - e^{1-t}\right) dt \\ &= \left(-t - e^{1-t}\Big|_0^1\right) + \left(t + e^{1-t}\Big|_1^3\right) \\ &= (-1 - 1 + e) + \left(3 + e^{-2} - 1 - 1\right) \\ &= e + e^{-2} - 1 \end{aligned}$$

$4 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{antidifferentiation} \\ 1 : \text{evaluation} \end{cases}$

OR

$x(t) = -t - e^{1-t}$

$x(0) = -e$

$x(1) = -2$

$x(3) = -e^{-2} - 3$

$$\begin{aligned} \text{Distance} &= (x(1) - x(0)) + (x(1) - x(3)) \\ &= (-2 + e) + \left(1 + e^{-2}\right) \\ &= e + e^{-2} - 1 \end{aligned}$$

OR

$4 : \begin{cases} 1 : \text{any antiderivative} \\ 1 : \text{evaluates } x(t) \text{ when } t = 0,\ 1,\ 3 \\ 1 : \text{evaluates distance between points} \\ 1 : \text{evaluates total distance} \end{cases}$

Question 2

A particle moves along the x-axis so that its velocity at time t is given by

$$v(t) = -(t+1)\sin\left(\frac{t^2}{2}\right).$$

At time $t = 0$, the particle is at position $x = 1$.

(a) Find the acceleration of the particle at time $t = 2$. Is the speed of the particle increasing at $t = 2$? Why or why not?

(b) Find all times t in the open interval $0 < t < 3$ when the particle changes direction. Justify your answer.

(c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.

(d) During the time interval $0 \le t \le 3$, what is the greatest distance between the particle and the origin? Show the work that leads to your answer.

Solution	Points
(a) $a(2) = v'(2) = 1.587$ or 1.588 $v(2) = -3\sin(2) < 0$ Speed is decreasing since $a(2) > 0$ and $v(2) < 0$.	2 : { 1 : $a(2)$ 1 : speed decreasing with reason
(b) $v(t) = 0$ when $\frac{t^2}{2} = \pi$ $t = \sqrt{2\pi}$ or 2.506 or 2.507 Since $v(t) < 0$ for $0 < t < \sqrt{2\pi}$ and $v(t) > 0$ for $\sqrt{2\pi} < t < 3$, the particle changes directions at $t = \sqrt{2\pi}$.	2 : { 1 : $t = \sqrt{2\pi}$ only 1 : justification
(c) Distance $= \int_0^3 \lvert v(t) \rvert \, dt = 4.333$ or 4.334	3 : { 1 : limits 1 : integrand 1 : answer
(d) $\int_0^{\sqrt{2\pi}} v(t)\,dt = -3.265$ $x(\sqrt{2\pi}) = x(0) + \int_0^{\sqrt{2\pi}} v(t)\,dt = -2.265$ Since the total distance from $t = 0$ to $t = 3$ is 4.334, the particle is still to the left of the origin at $t = 3$. Hence the greatest distance from the origin is 2.265.	2 : { 1 : $\pm$ (distance particle travels while velocity is negative) 1 : answer

Question 3

A particle moves along the y-axis so that its velocity v at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}\left(e^t\right)$. At time $t = 0$, the particle is at $y = -1$. (Note: $\tan^{-1} x = \arctan x$)

(a) Find the acceleration of the particle at time $t = 2$.

(b) Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

(c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

(d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

(a) $a(2) = v'(2) = -0.132$ or -0.133	1 : answer
(b) $v(2) = -0.436$ Speed is increasing since $a(2) < 0$ and $v(2) < 0$.	1 : answer with reason
(c) $v(t) = 0$ when $\tan^{-1}\left(e^t\right) = 1$ $t = \ln(\tan(1)) = 0.443$ is the only critical value for y. $v(t) > 0$ for $0 < t < \ln(\tan(1))$ $v(t) < 0$ for $t > \ln(\tan(1))$ $y(t)$ has an absolute maximum at $t = 0.443$.	3 : { 1 : sets $v(t) = 0$ 1 : identifies $t = 0.443$ as a candidate 1 : justifies absolute maximum
(d) $y(2) = -1 + \int_0^2 v(t)\,dt = -1.360$ or -1.361 The particle is moving away from the origin since $v(2) < 0$ and $y(2) < 0$.	4 : { 1 : $\int_0^2 v(t)\,dt$ 1 : handles initial condition 1 : value of $y(2)$ 1 : answer with reason

Question 3

A particle moves along the x-axis so that its velocity v at time t, for $0 \le t \le 5$, is given by $v(t) = \ln\left(t^2 - 3t + 3\right)$. The particle is at position $x = 8$ at time $t = 0$.

(a) Find the acceleration of the particle at time $t = 4$.

(b) Find all times t in the open interval $0 < t < 5$ at which the particle changes direction. During which time intervals, for $0 \le t \le 5$, does the particle travel to the left?

(c) Find the position of the particle at time $t = 2$.

(d) Find the average speed of the particle over the interval $0 \le t \le 2$.

(a) $a(4) = v'(4) = \dfrac{5}{7}$

1 : answer

(b) $v(t) = 0$

$t^2 - 3t + 3 = 1$

$t^2 - 3t + 2 = 0$

$(t-2)(t-1) = 0$

$t = 1, 2$

$v(t) > 0$ for $0 < t < 1$

$v(t) < 0$ for $1 < t < 2$

$v(t) > 0$ for $2 < t < 5$

The particle changes direction when $t = 1$ and $t = 2$.
The particle travels to the left when $1 < t < 2$.

3 :
- 1 : sets $v(t) = 0$
- 1 : direction change at $t = 1, 2$
- 1 : interval with reason

(c) $s(t) = s(0) + \int_0^t \ln\left(u^2 - 3u + 3\right)\,du$

$s(2) = 8 + \int_0^2 \ln\left(u^2 - 3u + 3\right)\,du$

$= 8.368$ or 8.369

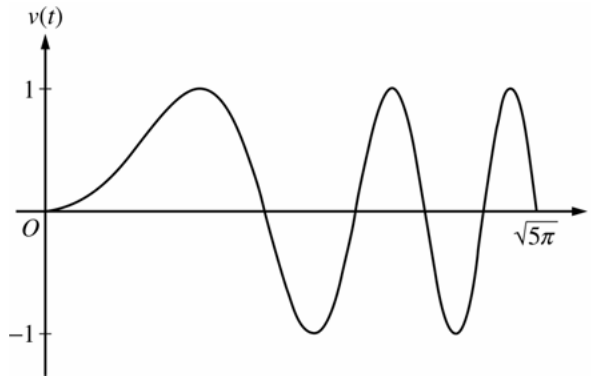
3 :
- 1 : $\int_0^2 \ln\left(u^2 - 3u + 3\right)\,du$
- 1 : handles initial condition
- 1 : answer

(d) $\dfrac{1}{2}\int_0^2 |v(t)|\,dt = 0.370$ or 0.371

2 :
- 1 : integral
- 1 : answer

Question 2

A particle moves along the x-axis so that its velocity v at time $t \geq 0$ is given by $v(t) = \sin\left(t^2\right)$. The graph of v is shown above for $0 \leq t \leq \sqrt{5\pi}$. The position of the particle at time t is $x(t)$ and its position at time $t = 0$ is $x(0) = 5$.



(a) Find the acceleration of the particle at time $t = 3$.

(b) Find the total distance traveled by the particle from time $t = 0$ to $t = 3$.

(c) Find the position of the particle at time $t = 3$.

(d) For $0 \leq t \leq \sqrt{5\pi}$, find the time t at which the particle is farthest to the right. Explain your answer.

(a) $a(3) = v'(3) = 6\cos 9 = -5.466$ or -5.467

1 : $a(3)$

(b) Distance $= \int_0^3 |v(t)|\, dt = 1.702$

OR

For $0 < t < 3$, $v(t) = 0$ when $t = \sqrt{\pi} = 1.77245$ and $t = \sqrt{2\pi} = 2.50663$

$x(0) = 5$

$x\left(\sqrt{\pi}\right) = 5 + \int_0^{\sqrt{\pi}} v(t)\, dt = 5.89483$

$x\left(\sqrt{2\pi}\right) = 5 + \int_0^{\sqrt{2\pi}} v(t)\, dt = 5.43041$

$x(3) = 5 + \int_0^3 v(t)\, dt = 5.77356$

$\left|x\left(\sqrt{\pi}\right) - x(0)\right| + \left|x\left(\sqrt{2\pi}\right) - x\left(\sqrt{\pi}\right)\right| + \left|x(3) - x\left(\sqrt{2\pi}\right)\right| = 1.702$

2 : $\begin{cases} 1 : \text{setup} \\ 1 : \text{answer} \end{cases}$

(c) $x(3) = 5 + \int_0^3 v(t)\, dt = 5.773$ or 5.774

3 : $\begin{cases} 2 \begin{cases} 1 : \text{integrand} \\ 1 : \text{uses } x(0) = 5 \end{cases} \\ 1 : \text{answer} \end{cases}$

(d) The particle's rightmost position occurs at time $t = \sqrt{\pi} = 1.772$.

The particle changes from moving right to moving left at those times t for which $v(t) = 0$ with $v(t)$ changing from positive to negative, namely at $t = \sqrt{\pi}, \sqrt{3\pi}, \sqrt{5\pi}$ $(t = 1.772, 3.070, 3.963)$.

Using $x(T) = 5 + \int_0^T v(t)\, dt$, the particle's positions at the times it changes from rightward to leftward movement are:

T:	0	$\sqrt{\pi}$	$\sqrt{3\pi}$	$\sqrt{5\pi}$
$x(T)$:	5	5.895	5.788	5.752

The particle is farthest to the right when $T = \sqrt{\pi}$.

3 : $\begin{cases} 1 : \text{sets } v(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

Question 4

A particle moves along the x-axis with position at time t given by $x(t) = e^{-t}\sin t$ for $0 \le t \le 2\pi$.

(a) Find the time t at which the particle is farthest to the left. Justify your answer.

(b) Find the value of the constant A for which $x(t)$ satisfies the equation $Ax''(t) + x'(t) + x(t) = 0$ for $0 < t < 2\pi$.

(a) $x'(t) = -e^{-t}\sin t + e^{-t}\cos t = e^{-t}(\cos t - \sin t)$

$x'(t) = 0$ when $\cos t = \sin t$. Therefore, $x'(t) = 0$ on $0 \le t \le 2\pi$ for $t = \dfrac{\pi}{4}$ and $t = \dfrac{5\pi}{4}$.

The candidates for the absolute minimum are at $t = 0, \dfrac{\pi}{4}, \dfrac{5\pi}{4}$, and 2π.

t	$x(t)$
0	$e^{0}\sin(0) = 0$
$\dfrac{\pi}{4}$	$e^{-\frac{\pi}{4}}\sin\left(\dfrac{\pi}{4}\right) > 0$
$\dfrac{5\pi}{4}$	$e^{-\frac{5\pi}{4}}\sin\left(\dfrac{5\pi}{4}\right) < 0$
2π	$e^{-2\pi}\sin(2\pi) = 0$

The particle is farthest to the left when $t = \dfrac{5\pi}{4}$.

5 :
- 2 : $x'(t)$
- 1 : sets $x'(t) = 0$
- 1 : answer
- 1 : justification

(b) $x''(t) = -e^{-t}(\cos t - \sin t) + e^{-t}(-\sin t - \cos t)$

$= -2e^{-t}\cos t$

$Ax''(t) + x'(t) + x(t)$

$= A\left(-2e^{-t}\cos t\right) + e^{-t}(\cos t - \sin t) + e^{-t}\sin t$

$= (-2A + 1)e^{-t}\cos t$

$= 0$

Therefore, $A = \dfrac{1}{2}$.

4 :
- 2 : $x''(t)$
- 1 : substitutes $x''(t)$, $x'(t)$, and $x(t)$ into $Ax''(t) + x'(t) + x(t)$
- 1 : answer

1989 AB3

A particle moves along the x-axis in such a way that its acceleration at time t for $t \geq 0$ is given by $a(t) = 4\cos(2t)$. At time $t = 0$, the velocity of the particle is $v(0) = 1$ and its position is $x(0) = 0$.

(a) Write an equation for the velocity $v(t)$ of the particle.

(b) Write an equation for the position $x(t)$ of the particle.

(c) For what values of t, $0 \leq t \leq \pi$, is the particle at rest?

1989 AB3
Solution

(a) $v(t) = \int 4\cos 2t\,dt$

$v(t) = 2\sin 2t + C$

$v(0) = 1 \Rightarrow C = 1$

$v(t) = 2\sin 2t + 1$

(b) $x(t) = \int 2\sin 2t + 1\ dt$

$x(t) = -\cos 2t + t + C$

$x(0) = 0 \Rightarrow C = 1$

$x(t) = -\cos 2t + t + 1$

(c) $2\sin 2t + 1 = 0$

$\sin 2t = -\frac{1}{2}$

$2t = \frac{7\pi}{6}, \frac{11\pi}{6}$

$t = \frac{7\pi}{12}, \frac{11\pi}{12}$

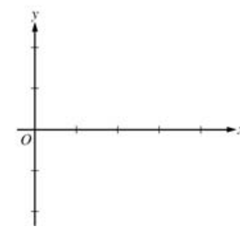
Question 4

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
$f(x)$	–1	Negative	0	Positive	2	Positive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	–3	Negative
$f''(x)$	–2	Negative	0	Positive	DNE	Negative	0	Positive

Let f be a function that is continuous on the interval $[0, 4)$. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of f do not exist at $x = 2$.

(a) For $0 < x < 4$, find all values of x at which f has a relative extremum. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

(b) On the axes provided, sketch the graph of a function that has all the characteristics of f.
(Note: Use the axes provided in the pink test booklet.)



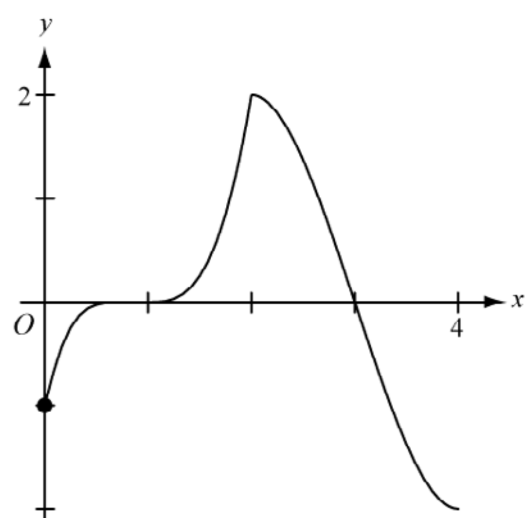
(c) Let g be the function defined by $g(x) = \int_1^x f(t)\,dt$ on the open interval $(0, 4)$. For $0 < x < 4$, find all values of x at which g has a relative extremum. Determine whether g has a relative maximum or a relative minimum at each of these values. Justify your answer.

(d) For the function g defined in part (c), find all values of x, for $0 < x < 4$, at which the graph of g has a point of inflection. Justify your answer.

(a) f has a relative maximum at $x = 2$ because f' changes from positive to negative at $x = 2$.

$2 : \begin{cases} 1 : \text{relative extremum at } x = 2 \\ 1 : \text{relative maximum with justification} \end{cases}$

(b)



$2 : \begin{cases} 1 : \text{points at } x = 0, 1, 2, 3 \text{ and behavior at } (2, 2) \\ 1 : \text{appropriate increasing/decreasing and concavity behavior} \end{cases}$

(c) $g'(x) = f(x) = 0$ at $x = 1, 3$.
g' changes from negative to positive at $x = 1$ so g has a relative minimum at $x = 1$. g' changes from positive to negative at $x = 3$ so g has a relative maximum at $x = 3$.

$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{critical points} \\ 1 : \text{answer with justification} \end{cases}$

(d) The graph of g has a point of inflection at $x = 2$ because $g'' = f'$ changes sign at $x = 2$.

$2 : \begin{cases} 1 : x = 2 \\ 1 : \text{answer with justification} \end{cases}$

Question 3

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	–4	4	2
4	–1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$
$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$
Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

$2 : \begin{cases} 1 : h(1) \text{ and } h(3) \\ 1 : \text{conclusion, using IVT} \end{cases}$

(b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$
Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.

$2 : \begin{cases} 1 : \dfrac{h(3) - h(1)}{3 - 1} \\ 1 : \text{conclusion, using MVT} \end{cases}$

(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$

$2 : \begin{cases} 1 : \text{apply chain rule} \\ 1 : \text{answer} \end{cases}$

(d) $g(1) = 2$, so $g^{-1}(2) = 1$.
$$\left(g^{-1}\right)'(2) = \frac{1}{g'\left(g^{-1}(2)\right)} = \frac{1}{g'(1)} = \frac{1}{5}$$
An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 2)$.

$3 : \begin{cases} 1 : g^{-1}(2) \\ 1 : \left(g^{-1}\right)'(2) \\ 1 : \text{tangent line equation} \end{cases}$

1991 AB5

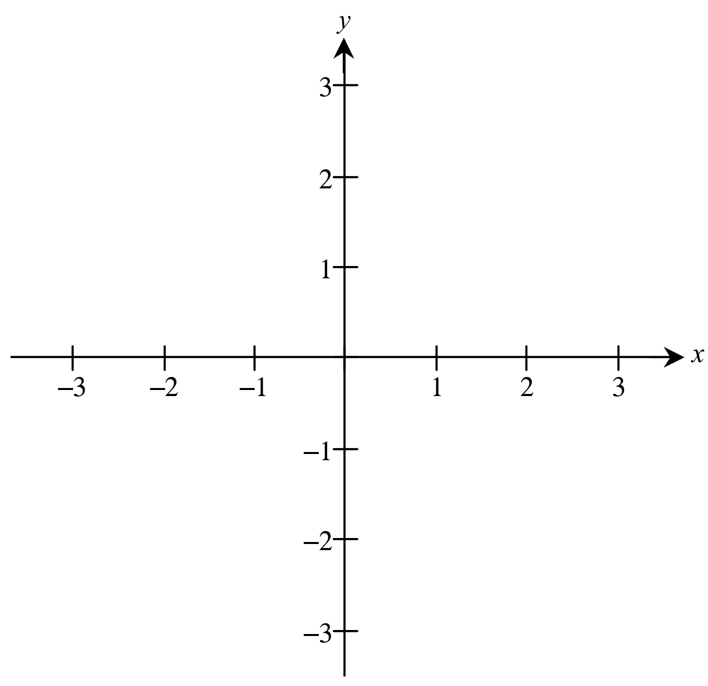
Let f be a function that is even and continuous on the closed interval $[-3,3]$. The function f and its derivatives have the properties indicated in the table below.

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$
$f(x)$	1	Positive	0	Negative	-1	Negative
$f'(x)$	Undefined	Negative	0	Negative	Undefined	Positive
$f''(x)$	Undefined	Positive	0	Negative	Undefined	Negative

(a) Find the x-coordinate of each point at which f attains an absolute maximum value or an absolute minimum value. For each x-coordinate you give, state whether f attains an absolute maximum or an absolute minimum.

(b) Find the x-coordinate of each point of inflection on the graph of f. Justify your answer.

(c) In the xy-plane provided below, sketch the graph of a function with all the given characteristics of f.

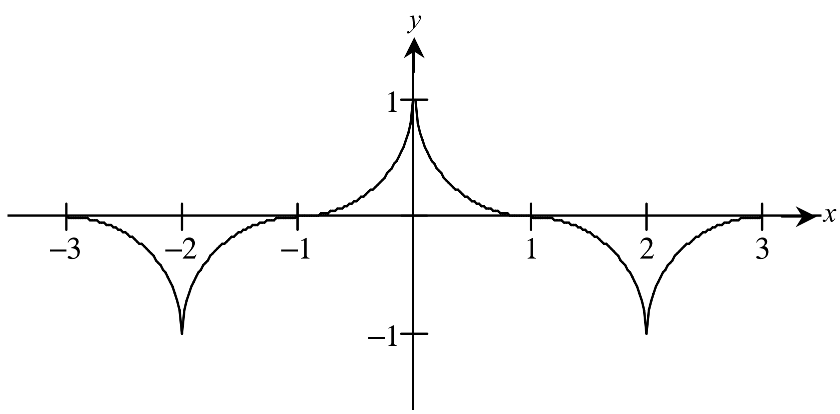


1991 AB5
Solution

(a) Absolute maximum at $x = 0$
Absolute minimum at $x = \pm 2$

(b) Points of inflection at $x = \pm 1$ because the sign of $f''(x)$ changes at $x = 1$ and f is even

(c)



Question 6

x	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
$f(x)$	-1	-4	-6	-7	-6	-4	-1
$f'(x)$	-7	-5	-3	0	3	5	7

Let f be a function that is differentiable for all real numbers. The table above gives the values of f and its derivative f' for selected points x in the closed interval $-1.5 \le x \le 1.5$. The second derivative of f has the property that $f''(x) > 0$ for $-1.5 \le x \le 1.5$.

(a) Evaluate $\int_0^{1.5} \left(3f'(x) + 4\right)dx$. Show the work that leads to your answer.

(b) Write an equation of the line tangent to the graph of f at the point where $x = 1$. Use this line to approximate the value of $f(1.2)$. Is this approximation greater than or less than the actual value of $f(1.2)$? Give a reason for your answer.

(c) Find a positive real number r having the property that there must exist a value c with $0 < c < 0.5$ and $f''(c) = r$. Give a reason for your answer.

(d) Let g be the function given by $g(x) = \begin{cases} 2x^2 - x - 7 & \text{for } x < 0 \\ 2x^2 + x - 7 & \text{for } x \ge 0. \end{cases}$

The graph of g passes through each of the points $\left(x, f(x)\right)$ given in the table above. Is it possible that f and g are the same function? Give a reason for your answer.

(a) $\int_0^{1.5} \left(3f'(x) + 4\right)dx = 3\int_0^{1.5} f'(x)\,dx + \int_0^{1.5} 4\,dx$

$= 3f(x) + 4x\Big|_0^{1.5} = 3(-1 - (-7)) + 4(1.5) = 24$

2 { 1 : antiderivative; 1 : answer }

(b) $y = 5(x - 1) - 4$

$f(1.2) \approx 5(0.2) - 4 = -3$

The approximation is less than $f(1.2)$ because the graph of f is concave up on the interval $1 < x < 1.2$.

3 { 1 : tangent line; 1 : computes y on tangent line at $x = 1.2$; 1 : answer with reason }

(c) By the Mean Value Theorem there is a c with $0 < c < 0.5$ such that

$$f''(c) = \frac{f'(0.5) - f'(0)}{0.5 - 0} = \frac{3 - 0}{0.5} = 6 = r$$

2 { 1 : reference to MVT for f' (or differentiability of f'); 1 : value of r for interval $0 \le x \le 0.5$ }

(d) $\lim_{x \to 0^-} g'(x) = \lim_{x \to 0^-} (4x - 1) = -1$

$\lim_{x \to 0^+} g'(x) = \lim_{x \to 0^+} (4x + 1) = +1$

Thus g' is not continuous at $x = 0$, but f' is continuous at $x = 0$, so $f \ne g$.

OR

$g''(x) = 4$ for all $x \ne 0$, but it was shown in part (c) that $f''(c) = 6$ for some $c \ne 0$, so $f \ne g$.

2 { 1 : answers "no" with reference to g' or g''; 1 : correct reason }

2. Let f be the function given by $f(x) = 2xe^{2x}$.

   (a) Find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

   (b) Find the absolute minimum value of f. Justify that your answer is an absolute minimum.

   (c) What is the range of f?

   (d) Consider the family of functions defined by $y = bxe^{bx}$, where b is a nonzero constant. Show that the absolute minimum value of bxe^{bx} is the same for all nonzero values of b.

(a) $\lim_{x\to-\infty} 2xe^{2x} = 0$

$\lim_{x\to\infty} 2xe^{2x} = \infty$ or DNE

$2 \begin{cases} 1\text{: } 0 \text{ as } x \to -\infty \\ 1\text{: } \infty \text{ or DNE as } x \to \infty \end{cases}$

(b) $f'(x) = 2e^{2x} + 2x \cdot 2 \cdot e^{2x} = 2e^{2x}(1+2x) = 0$

if $x = -1/2$

$f(-1/2) = -1/e$ or -0.368 or -0.367

$-1/e$ is an absolute minimum value because:

(i) $f'(x) < 0$ for all $x < -1/2$ and $f'(x) > 0$ for all $x > -1/2$

–or–

(ii) $f'(x)$: $-$ to the left of $-1/2$, $+$ to the right of $-1/2$

and $x = -1/2$ is the only critical number

3:
- 1: solves $f'(x) = 0$
- 1: evaluates f at student's critical point
  0/1 if not local minimum from student's derivative
- 1: justifies absolute minimum value
  0/1 for a local argument
  0/1 without explicit symbolic derivative

Note: 0/3 if no absolute minimum based on student's derivative

(c) Range of $f = [-1/e, \infty)$
or $[-0.367, \infty)$
or $[-0.368, \infty)$

1: answer

Note: must include the left–hand endpoint; exclude the right–hand "endpoint"

(d) $y' = be^{bx} + b^2xe^{bx} = be^{bx}(1+bx) = 0$

if $x = -1/b$

At $x = -1/b$, $y = -1/e$

y has an absolute minimum value of $-1/e$ for all nonzero b

3:
- 1: sets $y' = be^{bx}(1+bx) = 0$
- 1: solves student's $y' = 0$
- 1: evaluates y at a critical number and gets a value independent of b

Note: 0/3 if only considering specific values of b

Question 5

A cubic polynomial function f is defined by

$$f(x) = 4x^3 + ax^2 + bx + k$$

where a, b, and k are constants. The function f has a local minimum at $x = -1$, and the graph of f has a point of inflection at $x = -2$.

(a) Find the values of a and b.

(b) If $\int_0^1 f(x)\,dx = 32$, what is the value of k?

(a) $f'(x) = 12x^2 + 2ax + b$

$f''(x) = 24x + 2a$

$f'(-1) = 12 - 2a + b = 0$

$f''(-2) = -48 + 2a = 0$

$a = 24$

$b = -12 + 2a = 36$

$$5 : \begin{cases} 1 : f'(x) \\ 1 : f''(x) \\ 1 : f'(-1) = 0 \\ 1 : f''(-2) = 0 \\ 1 : a,\ b \end{cases}$$

(b) $\int_0^1 \left(4x^3 + 24x^2 + 36x + k\right)dx$

$= x^4 + 8x^3 + 18x^2 + kx\Big|_{x=0}^{x=1} = 27 + k$

$27 + k = 32$

$k = 5$

$$4 : \begin{cases} 2 : \text{antidifferentiation} \\ \quad < -1 > \text{ each error} \\ 1 : \text{expression in } k \\ 1 : k \end{cases}$$

Question 4

Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of h concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of h at $x = 4$.

(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

(a) $h'(x) = 0$ at $x = \pm\sqrt{2}$

$h'(x)$	$-$	0	$+$	und	$-$	0	$+$
x		$-\sqrt{2}$		0		$\sqrt{2}$	

Local minima at $x = -\sqrt{2}$ and at $x = \sqrt{2}$

4 :
- 1 : $x = \pm\sqrt{2}$
- 1 : analysis
- 2 : conclusions
  $< -1 >$ not dealing with discontinuity at 0

(b) $h''(x) = 1 + \dfrac{2}{x^2} > 0$ for all $x \neq 0$. Therefore, the graph of h is concave up for all $x \neq 0$.

3 :
- 1 : $h''(x)$
- 1 : $h''(x) > 0$
- 1 : answer

(c) $h'(4) = \dfrac{16 - 2}{4} = \dfrac{7}{2}$

$y + 3 = \dfrac{7}{2}(x - 4)$

1 : tangent line equation

(d) The tangent line is below the graph because the graph of h is concave up for $x > 4$.

1 : answer with reason

Question 6

Let f be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \le x \le 3 \\ 5 - x & \text{for } 3 < x \le 5. \end{cases}$$

(a) Is f continuous at $x = 3$? Explain why or why not.

(b) Find the average value of $f(x)$ on the closed interval $0 \le x \le 5$.

(c) Suppose the function g is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases}$$

where k and m are constants. If g is differentiable at $x = 3$, what are the values of k and m?

(a) f is continuous at $x = 3$ because

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = 2.$

Therefore, $\lim\limits_{x \to 3} f(x) = 2 = f(3).$

$2 : \begin{cases} 1 : \text{answers "yes" and equates the values of the left- and right-hand limits} \\ 1 : \text{explanation involving limits} \end{cases}$

(b) $$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$$
$$= \frac{2}{3}(x+1)^{3/2}\Big|_0^3 + \left(5x - \frac{1}{2}x^2\right)\Big|_3^5$$
$$= \left(\frac{16}{3} - \frac{2}{3}\right) + \left(\frac{25}{2} - \frac{21}{2}\right) = \frac{20}{3}$$

Average value: $\dfrac{1}{5}\displaystyle\int_0^5 f(x)\,dx = \dfrac{4}{3}$

$4 : \begin{cases} 1 : k\int_0^3 f(x)\,dx + k\int_3^5 f(x)\,dx \\ \quad (\text{where } k \ne 0) \\ 1 : \text{antiderivative of } \sqrt{x+1} \\ 1 : \text{antiderivative of } 5 - x \\ 1 : \text{evaluation and answer} \end{cases}$

(c) Since g is continuous at $x = 3$, $2k = 3m + 2$.

$$g'(x) = \begin{cases} \dfrac{k}{2\sqrt{x+1}} & \text{for } 0 < x < 3 \\ m & \text{for } 3 < x < 5 \end{cases}$$

$\lim\limits_{x \to 3^-} g'(x) = \dfrac{k}{4}$ and $\lim\limits_{x \to 3^+} g'(x) = m$

Since these two limits exist and g is differentiable at $x = 3$, the two limits are equal. Thus $\dfrac{k}{4} = m$.

$8m = 3m + 2$; $m = \dfrac{2}{5}$ and $k = \dfrac{8}{5}$

$3 : \begin{cases} 1 : 2k = 3m + 2 \\ 1 : \dfrac{k}{4} = m \\ 1 : \text{values for } k \text{ and } m \end{cases}$

Question 6

Let f be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0,$ where k is a positive constant.

(a) Find $f'(x)$ and $f''(x)$.

(b) For what value of the constant k does f have a critical point at $x = 1$? For this value of k, determine whether f has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

(c) For a certain value of the constant k, the graph of f has a point of inflection on the x-axis. Find this value of k.

(a) $f'(x) = \dfrac{k}{2\sqrt{x}} - \dfrac{1}{x}$

$f''(x) = -\dfrac{1}{4}kx^{-3/2} + x^{-2}$

$2 : \begin{cases} 1 : f'(x) \\ 1 : f''(x) \end{cases}$

(b) $f'(1) = \dfrac{1}{2}k - 1 = 0 \Rightarrow k = 2$

When $k = 2$, $f'(1) = 0$ and $f''(1) = -\dfrac{1}{2} + 1 > 0.$

f has a relative minimum value at $x = 1$ by the Second Derivative Test.

$4 : \begin{cases} 1 : \text{sets } f'(1) = 0 \text{ or } f'(x) = 0 \\ 1 : \text{solves for } k \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(c) At this inflection point, $f''(x) = 0$ and $f(x) = 0.$

$f''(x) = 0 \Rightarrow \dfrac{-k}{4x^{3/2}} + \dfrac{1}{x^2} = 0 \Rightarrow k = \dfrac{4}{\sqrt{x}}$

$f(x) = 0 \Rightarrow k\sqrt{x} - \ln x = 0 \Rightarrow k = \dfrac{\ln x}{\sqrt{x}}$

Therefore, $\dfrac{4}{\sqrt{x}} = \dfrac{\ln x}{\sqrt{x}}$

$\Rightarrow 4 = \ln x$

$\Rightarrow x = e^4$

$\Rightarrow k = \dfrac{4}{e^2}$

$3 : \begin{cases} 1 : f''(x) = 0 \text{ or } f(x) = 0 \\ 1 : \text{equation in one variable} \\ 1 : \text{answer} \end{cases}$

Question 6

Let f be the function given by $f(x) = \dfrac{\ln x}{x}$ for all $x > 0$. The derivative of f is given by $f'(x) = \dfrac{1 - \ln x}{x^2}$.

(a) Write an equation for the line tangent to the graph of f at $x = e^2$.

(b) Find the x-coordinate of the critical point of f. Determine whether this point is a relative minimum, a relative maximum, or neither for the function f. Justify your answer.

(c) The graph of the function f has exactly one point of inflection. Find the x-coordinate of this point.

(d) Find $\lim\limits_{x \to 0^+} f(x)$.

(a) $f(e^2) = \dfrac{\ln e^2}{e^2} = \dfrac{2}{e^2}, \quad f'(e^2) = \dfrac{1 - \ln e^2}{(e^2)^2} = -\dfrac{1}{e^4}$

An equation for the tangent line is $y = \dfrac{2}{e^2} - \dfrac{1}{e^4}(x - e^2)$.

$2 : \begin{cases} 1 : f(e^2) \text{ and } f'(e^2) \\ 1 : \text{answer} \end{cases}$

(b) $f'(x) = 0$ when $x = e$. The function f has a relative maximum at $x = e$ because $f'(x)$ changes from positive to negative at $x = e$.

$3 : \begin{cases} 1 : x = e \\ 1 : \text{relative maximum} \\ 1 : \text{justification} \end{cases}$

(c) $f''(x) = \dfrac{-\frac{1}{x}x^2 - (1 - \ln x)2x}{x^4} = \dfrac{-3 + 2\ln x}{x^3}$ for all $x > 0$

$f''(x) = 0$ when $-3 + 2\ln x = 0$

$x = e^{3/2}$

The graph of f has a point of inflection at $x = e^{3/2}$ because $f''(x)$ changes sign at $x = e^{3/2}$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer} \end{cases}$

(d) $\lim\limits_{x \to 0^+} \dfrac{\ln x}{x} = -\infty$ or Does Not Exist

1 : answer

4. Suppose that the function f has a continuous second derivative for all x, and that $f(0) = 2$, $f'(0) = -3$, and $f''(0) = 0$. Let g be a function whose derivative is given by $g'(x) = e^{-2x}(3f(x) + 2f'(x))$ for all x.
   (a) Write an equation of the line tangent to the graph of f at the point where $x = 0$.
   (b) Is there sufficient information to determine whether or not the graph of f has a point of inflection when $x = 0$? Explain your answer.
   (c) Given that $g(0) = 4$, write an equation of the line tangent to the graph of g at the point where $x = 0$.
   (d) Show that $g''(x) = e^{-2x}(-6f(x) - f'(x) + 2f''(x))$. Does g have a local maximum at $x = 0$? Justify your answer.

(a) Slope at $x = 0$ is $f'(0) = -3$

At $x = 0$, $y = 2$

$y - 2 = -3(x - 0)$

1: equation

(b) No. Whether $f''(x)$ changes sign at $x = 0$ is unknown. The only given value of $f''(x)$ is $f''(0) = 0$.

2 { 1: answer; 1: explanation }

(c) $g'(x) = e^{-2x}(3f(x) + 2f'(x))$

$$g'(0) = e^0(3f(0) + 2f'(0)) = 3(2) + 2(-3) = 0$$

$y - 4 = 0(x - 0)$

$y = 4$

2 { 1: $g'(0)$; 1: equation }

(d) $g'(x) = e^{-2x}(3f(x) + 2f'(x))$

$$g''(x) = (-2e^{-2x})(3f(x) + 2f'(x)) + e^{-2x}(3f'(x) + 2f''(x)) = e^{-2x}(-6f(x) - f'(x) + 2f''(x))$$

$g''(0) = e^0[(-6)(2) - (-3) + 2(0)] = -9$

Since $g'(0) = 0$ and $g''(0) < 0$, g does have a local maximum at $x = 0$.

4 {
- 2: verify derivative
  - 0/2 product or chain rule error
  - <−1> algebra errors
- 1: $g'(0) = 0$ and $g''(0)$
- 1: answer and reasoning
}

Question 6

Let f be the function satisfying $f'(x) = x\sqrt{f(x)}$ for all real numbers x, where $f(3) = 25$.

(a) Find $f''(3)$.

(b) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = x\sqrt{y}$ with the initial condition $f(3) = 25$.

(a) $f''(x) = \sqrt{f(x)} + x \cdot \dfrac{f'(x)}{2\sqrt{f(x)}} = \sqrt{f(x)} + \dfrac{x^2}{2}$

$f''(3) = \sqrt{25} + \dfrac{9}{2} = \dfrac{19}{2}$

$3 : \begin{cases} 2 : f''(x) \\ \quad < -2 > \text{ product or} \\ \quad \text{chain rule error} \\ 1 : \text{value at } x = 3 \end{cases}$

(b) $\dfrac{1}{\sqrt{y}}\,dy = x\,dx$

$2\sqrt{y} = \dfrac{1}{2}x^2 + C$

$2\sqrt{25} = \dfrac{1}{2}(3)^2 + C\ ;\ C = \dfrac{11}{2}$

$\sqrt{y} = \dfrac{1}{4}x^2 + \dfrac{11}{4}$

$y = \left(\dfrac{1}{4}x^2 + \dfrac{11}{4}\right)^2 = \dfrac{1}{16}\left(x^2 + 11\right)^2$

$6 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivative of } dy \text{ term} \\ 1 : \text{antiderivative of } dx \text{ term} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } f(3) = 25 \\ 1 : \text{solves for } y \end{cases}$

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Question 6

The twice-differentiable function f is defined for all real numbers and satisfies the following conditions:

$$f(0) = 2, \quad f'(0) = -4, \text{ and } f''(0) = 3.$$

(a) The function g is given by $g(x) = e^{ax} + f(x)$ for all real numbers, where a is a constant. Find $g'(0)$ and $g''(0)$ in terms of a. Show the work that leads to your answers.

(b) The function h is given by $h(x) = \cos(kx) f(x)$ for all real numbers, where k is a constant. Find $h'(x)$ and write an equation for the line tangent to the graph of h at $x = 0$.

(a) $g'(x) = ae^{ax} + f'(x)$

$g'(0) = a - 4$

$g''(x) = a^2 e^{ax} + f''(x)$

$g''(0) = a^2 + 3$

$$4 : \begin{cases} 1 : g'(x) \\ 1 : g'(0) \\ 1 : g''(x) \\ 1 : g''(0) \end{cases}$$

(b) $h'(x) = f'(x)\cos(kx) - k\sin(kx) f(x)$

$h'(0) = f'(0)\cos(0) - k\sin(0) f(0) = f'(0) = -4$

$h(0) = \cos(0) f(0) = 2$

The equation of the tangent line is $y = -4x + 2$.

$$5 : \begin{cases} 2 : h'(x) \\ 3 : \begin{cases} 1 : h'(0) \\ 1 : h(0) \\ 1 : \text{equation of tangent line} \end{cases} \end{cases}$$

Question 6

Let f be a twice-differentiable function such that $f(2) = 5$ and $f(5) = 2$. Let g be the function given by $g(x) = f(f(x))$.

(a) Explain why there must be a value c for $2 < c < 5$ such that $f'(c) = -1$.

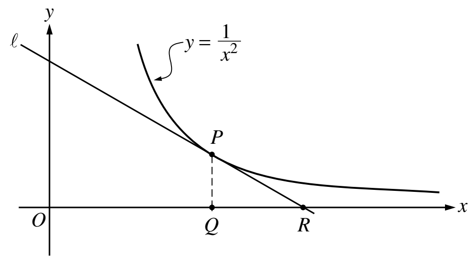
(b) Show that $g'(2) = g'(5)$. Use this result to explain why there must be a value k for $2 < k < 5$ such that $g''(k) = 0$.

(c) Show that if $f''(x) = 0$ for all x, then the graph of g does not have a point of inflection.

(d) Let $h(x) = f(x) - x$. Explain why there must be a value r for $2 < r < 5$ such that $h(r) = 0$.

(a) The Mean Value Theorem guarantees that there is a value c, with $2 < c < 5$, so that $f'(c) = \frac{f(5) - f(2)}{5 - 2} = \frac{2 - 5}{5 - 2} = -1.$	$2 : \begin{cases} 1 : \frac{f(5) - f(2)}{5 - 2} \\ 1 : \text{conclusion, using MVT} \end{cases}$
(b) $g'(x) = f'(f(x)) \cdot f'(x)$ $g'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2)$ $g'(5) = f'(f(5)) \cdot f'(5) = f'(2) \cdot f'(5)$ Thus, $g'(2) = g'(5)$. Since f is twice-differentiable, g' is differentiable everywhere, so the Mean Value Theorem applied to g' on $[2, 5]$ guarantees there is a value k, with $2 < k < 5$, such that $g''(k) = \frac{g'(5) - g'(2)}{5 - 2} = 0.$	$3 : \begin{cases} 1 : g'(x) \\ 1 : g'(2) = f'(5) \cdot f'(2) = g'(5) \\ 1 : \text{uses MVT with } g' \end{cases}$
(c) $g''(x) = f''(f(x)) \cdot f'(x) \cdot f'(x) + f'(f(x)) \cdot f''(x)$ If $f''(x) = 0$ for all x, then $g''(x) = 0 \cdot f'(x) \cdot f'(x) + f'(f(x)) \cdot 0 = 0$ for all x. Thus, there is no x-value at which $g''(x)$ changes sign, so the graph of g has no inflection points. OR If $f''(x) = 0$ for all x, then f is linear, so $g = f \circ f$ is linear and the graph of g has no inflection points.	$2 : \begin{cases} 1 : \text{considers } g'' \\ 1 : g''(x) = 0 \text{ for all } x \end{cases}$ OR $2 : \begin{cases} 1 : f \text{ is linear} \\ 1 : g \text{ is linear} \end{cases}$
(d) Let $h(x) = f(x) - x$. $h(2) = f(2) - 2 = 5 - 2 = 3$ $h(5) = f(5) - 5 = 2 - 5 = -3$ Since $h(2) > 0 > h(5)$, the Intermediate Value Theorem guarantees that there is a value r, with $2 < r < 5$, such that $h(r) = 0$.	$2 : \begin{cases} 1 : h(2) \text{ and } h(5) \\ 1 : \text{conclusion, using IVT} \end{cases}$

6. In the figure above, line ℓ is tangent to the graph of $y = \frac{1}{x^2}$ at point P, with coordinates $\left(w, \frac{1}{w^2}\right)$, where $w > 0$. Point Q has coordinates $(w, 0)$. Line ℓ crosses the x–axis at the point R, with coordinates $(k, 0)$.



(a) Find the value of k when $w = 3$.

(b) For all $w > 0$, find k in terms of w.

(c) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of k with respect to time?

(d) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of the area of $\triangle PQR$ with respect to time? Determine whether the area is increasing or decreasing at this instant.

(a) $\frac{dy}{dx} = -\frac{2}{x^3}$; $\left.\frac{dy}{dx}\right|_{x=3} = -\frac{2}{27}$

Line ℓ through $\left(3, \frac{1}{9}\right)$ and $(k, 0)$ has slope $-\frac{2}{27}$.

Therefore, $\frac{0 - \frac{1}{9}}{k - 3} = -\frac{2}{27}$ or $0 - \frac{1}{9} = -\frac{2}{27}(k - 3)$

$k = \frac{9}{2}$

$$2 \begin{cases} 1\text{: } \left.\frac{dy}{dx}\right|_{x=3} \\ 1\text{: answer} \end{cases}$$

(b) Line ℓ through $\left(w, \frac{1}{w^2}\right)$ and $(k, 0)$ has slope $-\frac{2}{w^3}$.

Therefore, $\frac{0 - \frac{1}{w^2}}{k - w} = -\frac{2}{w^3}$ or $0 - \frac{1}{w^2} = -\frac{2}{w^3}(k - w)$

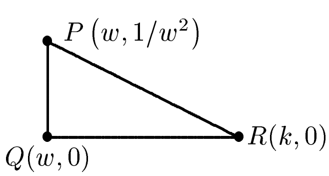
$k = \frac{3}{2}w$

$$2 \begin{cases} 1\text{: equation relating } w \text{ and } k\text{, using slopes} \\ 1\text{: answer} \end{cases}$$

(c) $\frac{dk}{dt} = \frac{3}{2}\frac{dw}{dt} = \frac{3}{2} \cdot 7 = \frac{21}{2}$; $\left.\frac{dk}{dt}\right|_{w=5} = \frac{21}{2}$

1: answer using $\frac{dw}{dt} = 7$

(d)



$A = \frac{1}{2}(k - w)\frac{1}{w^2} = \frac{1}{2}\left(\frac{3}{2}w - w\right)\frac{1}{w^2} = \frac{1}{4w}$

$\frac{dA}{dt} = -\frac{1}{4w^2}\frac{dw}{dt}$

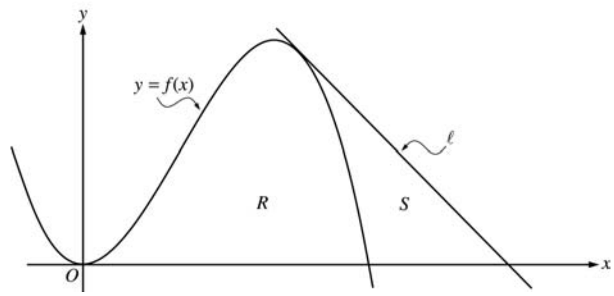
$\left.\frac{dA}{dt}\right|_{w=5} = -\frac{1}{100} \cdot 7 = -0.07$

Therefore, area is decreasing.

$$4 \begin{cases} 1\text{: area in terms of } w \text{ and/or } k \\ 1\text{: } \frac{dA}{dt} \text{ implicitly} \\ 1\text{: } \left.\frac{dA}{dt}\right|_{w=5} \text{ using } \frac{dw}{dt} = 7 \\ 1\text{: conclusion} \end{cases}$$

Note: 0/4 if A constant

Question 1



Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.

(a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of S.

(c) Find the volume of the solid generated when R is revolved about the x-axis.

(a) $f'(x) = 8x - 3x^2$; $f'(3) = 24 - 27 = -3$

$f(3) = 36 - 27 = 9$

Tangent line at $x = 3$ is

$y = -3(x - 3) + 9 = -3x + 18,$

which is the equation of line ℓ.

2 :
- 1 : finds $f'(3)$ and $f(3)$
- 1 : finds equation of tangent line
  or
  shows (3,9) is on both the graph of f and line ℓ

(b) $f(x) = 0$ at $x = 4$

The line intersects the x-axis at $x = 6$.

$$\text{Area} = \frac{1}{2}(3)(9) - \int_3^4 \left(4x^2 - x^3\right) dx$$

$= 7.916$ or 7.917

OR

$$\text{Area} = \int_3^4 \left((18 - 3x) - \left(4x^2 - x^3\right)\right) dx + \frac{1}{2}(2)(18 - 12)$$

$= 7.916$ or 7.917

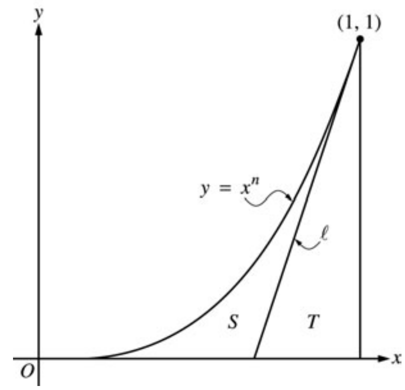
4 :
- 2 : integral for non-triangular region
  - 1 : limits
  - 1 : integrand
- 1 : area of triangular region
- 1 : answer

(c) $$\text{Volume} = \pi \int_0^4 \left(4x^2 - x^3\right)^2 dx$$

$= 156.038\pi$ or 490.208

3 :
- 1 : limits and constant
- 1 : integrand
- 1 : answer

Question 6



Let ℓ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown above.

(a) Find $\int_0^1 x^n\,dx$ in terms of n.

(b) Let T be the triangular region bounded by ℓ, the x-axis, and the line $x = 1$. Show that the area of T is $\frac{1}{2n}$.

(c) Let S be the region bounded by the graph of $y = x^n$, the line ℓ, and the x-axis. Express the area of S in terms of n and determine the value of n that maximizes the area of S.

(a) $\int_0^1 x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$

2 : $\begin{cases} 1 : \text{antiderivative of } x^n \\ 1 : \text{answer} \end{cases}$

(b) Let b be the length of the base of triangle T.

$\frac{1}{b}$ is the slope of line ℓ, which is n

$\text{Area}(T) = \frac{1}{2}b(1) = \frac{1}{2n}$

3 : $\begin{cases} 1 : \text{slope of line } \ell \text{ is } n \\ 1 : \text{base of } T \text{ is } \frac{1}{n} \\ 1 : \text{shows area is } \frac{1}{2n} \end{cases}$

(c) $\text{Area}(S) = \int_0^1 x^n\,dx - \text{Area}(T)$

$= \frac{1}{n+1} - \frac{1}{2n}$

$\frac{d}{dn}\text{Area}(S) = -\frac{1}{(n+1)^2} + \frac{1}{2n^2} = 0$

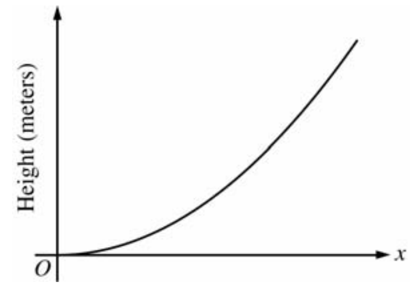
$2n^2 = (n+1)^2$

$\sqrt{2}\,n = (n+1)$

$n = \frac{1}{\sqrt{2}-1} = 1 + \sqrt{2}$

4 : $\begin{cases} 1 : \text{area of } S \text{ in terms of } n \\ 1 : \text{derivative} \\ 1 : \text{sets derivative equal to 0} \\ 1 : \text{solves for } n \end{cases}$

Question 3



The figure above is the graph of a function of x, which models the height of a skateboard ramp. The function meets the following requirements.

(i) At $x = 0$, the value of the function is 0, and the slope of the graph of the function is 0.
(ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.
(iii) Between $x = 0$ and $x = 4$, the function is increasing.

(a) Let $f(x) = ax^2$, where a is a nonzero constant. Show that it is not possible to find a value for a so that f meets requirement (ii) above.

(b) Let $g(x) = cx^3 - \frac{x^2}{16}$, where c is a nonzero constant. Find the value of c so that g meets requirement (ii) above. Show the work that leads to your answer.

(c) Using the function g and your value of c from part (b), show that g does not meet requirement (iii) above.

(d) Let $h(x) = \frac{x^n}{k}$, where k is a nonzero constant and n is a positive integer. Find the values of k and n so that h meets requirement (ii) above. Show that h also meets requirements (i) and (iii) above.

(a) $f(4) = 1$ implies that $a = \frac{1}{16}$ and $f'(4) = 2a(4) = 1$ implies that $a = \frac{1}{8}$. Thus, f cannot satisfy (ii).

$2 : \begin{cases} 1 : a = \frac{1}{16} \text{ or } a = \frac{1}{8} \\ 1 : \text{shows } a \text{ does not work} \end{cases}$

(b) $g(4) = 64c - 1 = 1$ implies that $c = \frac{1}{32}$.

When $c = \frac{1}{32}$, $g'(4) = 3c(4)^2 - \frac{2(4)}{16} = 3\left(\frac{1}{32}\right)(16) - \frac{1}{2} = 1$

$1 :$ value of c

(c) $g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = \frac{1}{32}x(3x - 4)$

$g'(x) < 0$ for $0 < x < \frac{4}{3}$, so g does not satisfy (iii).

$2 : \begin{cases} 1 : g'(x) \\ 1 : \text{explanation} \end{cases}$

(d) $h(4) = \frac{4^n}{k} = 1$ implies that $4^n = k$.

$h'(4) = \frac{n4^{n-1}}{k} = \frac{n4^{n-1}}{4^n} = \frac{n}{4} = 1$ gives $n = 4$ and $k = 4^4 = 256$.

$h(x) = \frac{x^4}{256} \Rightarrow h(0) = 0.$

$h'(x) = \frac{4x^3}{256} \Rightarrow h'(0) = 0$ and $h'(x) > 0$ for $0 < x < 4$.

$4 : \begin{cases} 1 : \frac{4^n}{k} = 1 \\ 1 : \frac{n4^{n-1}}{k} = 1 \\ 1 : \text{values for } k \text{ and } n \\ 1 : \text{verifications} \end{cases}$

Question 4

The functions f and g are given by $f(x) = \int_0^{3x} \sqrt{4 + t^2}\, dt$ and $g(x) = f(\sin x)$.

(a) Find $f'(x)$ and $g'(x)$.

(b) Write an equation for the line tangent to the graph of $y = g(x)$ at $x = \pi$.

(c) Write, but do not evaluate, an integral expression that represents the maximum value of g on the interval $0 \le x \le \pi$. Justify your answer.

(a) $f'(x) = 3\sqrt{4 + (3x)^2}$

$g'(x) = f'(\sin x) \cdot \cos x$

$= 3\sqrt{4 + (3\sin x)^2} \cdot \cos x$

$4 : \begin{cases} 2 : f'(x) \\ 2 : g'(x) \end{cases}$

(b) $g(\pi) = 0,\ g'(\pi) = -6$

Tangent line: $y = -6(x - \pi)$

$2 : \begin{cases} 1 : g(\pi) \text{ or } g'(\pi) \\ 1 : \text{tangent line equation} \end{cases}$

(c) For $0 < x < \pi$, $g'(x) = 0$ only at $x = \dfrac{\pi}{2}$.

$g(0) = g(\pi) = 0$

$g\left(\dfrac{\pi}{2}\right) = \int_0^3 \sqrt{4 + t^2}\, dt > 0$

The maximum value of g on $[0, \pi]$ is

$\int_0^3 \sqrt{4 + t^2}\, dt.$

$3 : \begin{cases} 1 : \text{sets } g'(x) = 0 \\ 1 : \text{justifies maximum at } \dfrac{\pi}{2} \\ 1 : \text{integral expression for } g\left(\dfrac{\pi}{2}\right) \end{cases}$